Good Afternoon.

Advanced algorithms and data structures

Lecture 1: Max Flow 1

Jacob Holm (jaho@di.ku.dk)

November 18th 2024

Today's Lecture

Introduction to AADS

This course is mostly about algorithms and how to analyse them.

We want *efficient* solutions when possible, where the meaning of "efficient" may depend on the problem.

In particular, we will focus on Polynomial time algorithms (fast) versus Exponential time algorithms (slow).

This weeks topic has a polynomial time algorithm. Later we will touch on some problems where we don't know or expect polynomial time algorithms to exist, and some ways to deal with that.

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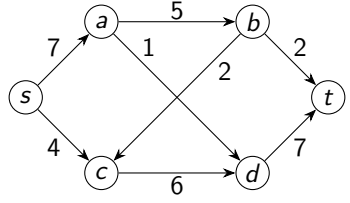
Flow network



Definition

A *flow network* consists of a directed graph $G = (V, E)$, a source $s \in V$, a sink $t \in V \setminus \{s\}$, and a capacity function $c : V \times V \to \mathbb{R}$ such that

- $c(u, v) \geq 0$ for all $u, v \in V$, and
- if $(u, v) \notin E$ then $c(u, v) = 0$

We will assume that G has no self-loops and no antiparallel edges.

Example of a flow network.
Graph with nodes s and t. Send goods/data/water from s to t. Can not accumulate in intermediate nodes, so what goes in must come out (flow conservation).
Capacities, capacity constraint.
Max flow can be used by itself, or as black box for solving other problems.

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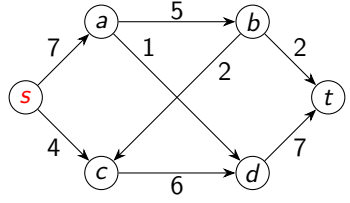

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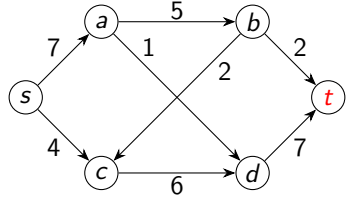

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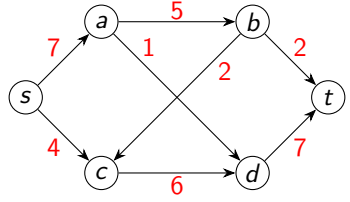

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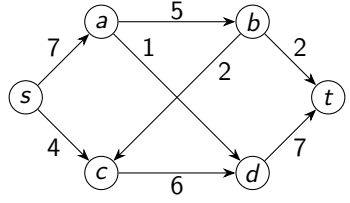

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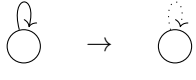

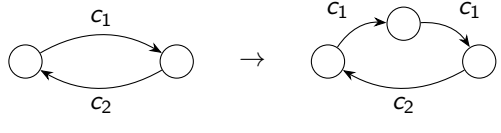

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Capacities, capacity constraint.
Max flow can be used by itself, or as black box for solving other problems.
No self-loops or antiparallel edges allowed by our algorithms/theorems. This is without loss of generality as we can always get rid of them.

Flow and max-Flow

Definition

A flow in flow network (G, s, t, c) is a function $f : V \times V \to \mathbb{R}$ such that:

1. $\forall u, v \in V$: $0 \leq f(u, v) \leq c(u, v)$ (capacity constraints)
2. $\forall v \in V \setminus \{s, t\}$: $\sum_{u \in V} f(u, v) = \sum_{w \in V} f(v, w)$ (flow conservation)

   Equivalently: $\sum_{(u,v) \in E} f(u, v) = \sum_{(v,w) \in E} f(v, w)$. Why?

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The value $|f|$ of a flow f is defined as:

$$|f| := \sum_{v \in V} f(s, v) - \sum_{u \in V} f(u, s) = \sum_{v \in V} (f(s, v) - f(v, s))$$

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A max-flow is a flow of maximum value.

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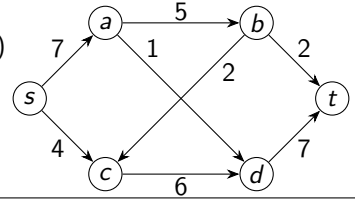
Our goal in the max-flow problem is to compute such a max-flow.

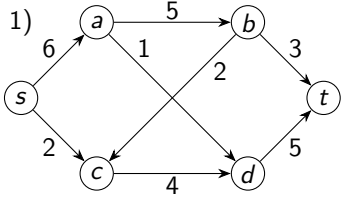
Examples: Which of these are flows? What are the values?

Flow network
(capacities on edges)



Candidate flows (flow on edges)

1)



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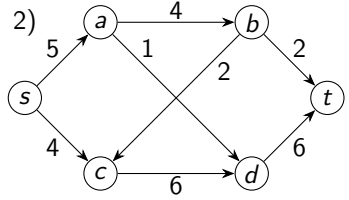

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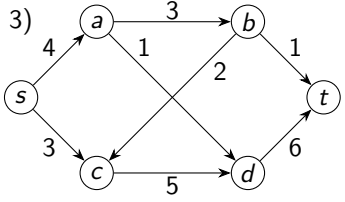

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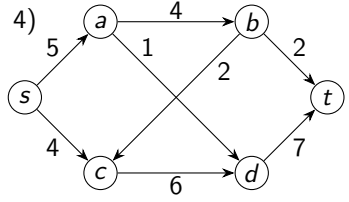

Ex 1: No. Capacity violation at (b, t).
Ex 2: No. Flow conservation violation at d.
Ex 3: Yes. Value 7
Ex 4: Yes. Value 9. Actually a max flow.

Ford-Fulkerson Method (informal)

Example 1:
Example 2: cancelling flow

function FORD-FULKERSON($G = (V, E)$, s, t, c)
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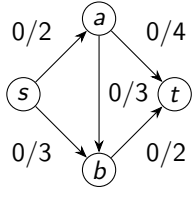

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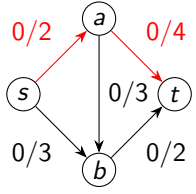

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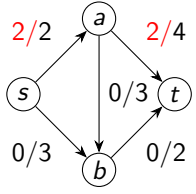

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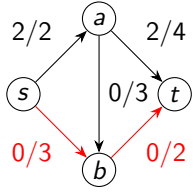

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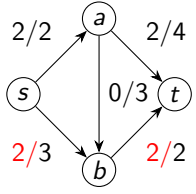

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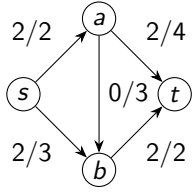

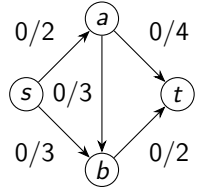

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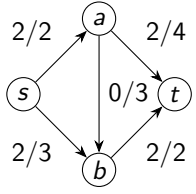

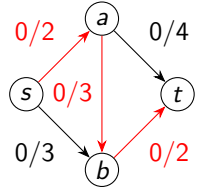

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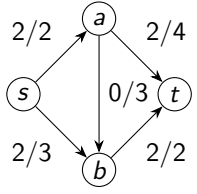

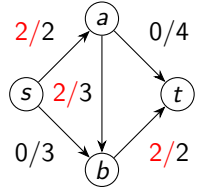

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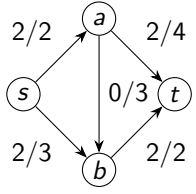

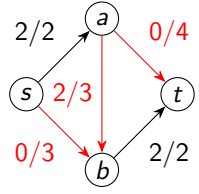

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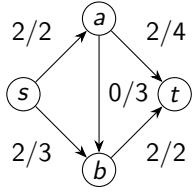

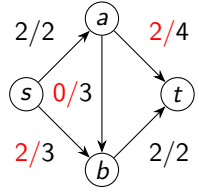

Break

Residual network

Recall that G has no self-loops or anti-parallel edges.

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Example: What are the edges and residual capacities.

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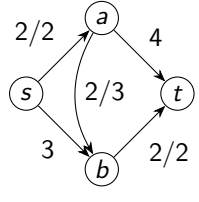

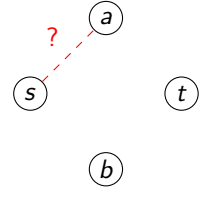

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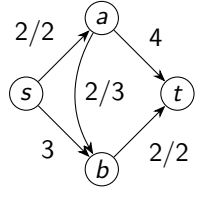

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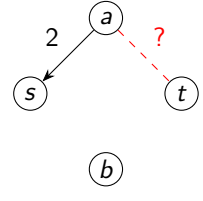

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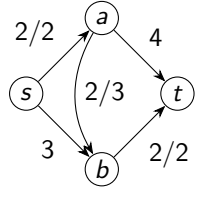

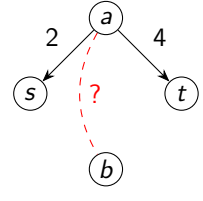

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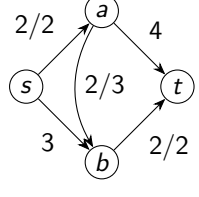

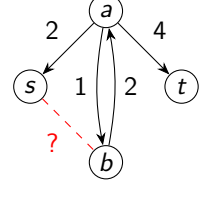

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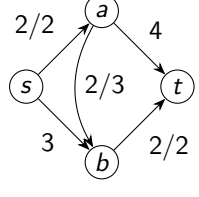

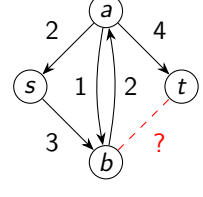

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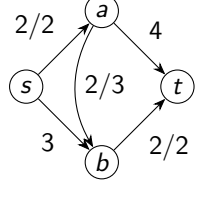

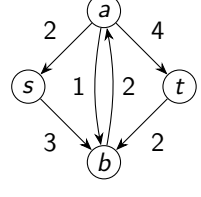

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function FORD-FULKERSON($G = (V, E)$, s, t, c)
  $f \leftarrow 0$
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What is the max flow along a path?

Illustrate "max flow along p".
$f \uparrow f'$ defined later.
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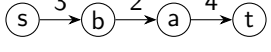

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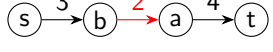

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$f \uparrow f'$ is a flow in G of value $|f \uparrow f'| = |f| + |f'|$.

Proof of flow conservation.

Let $u \in V \setminus \{s,t\}$, then

$$\sum_{(u,v)\in E} (f \uparrow f')(u,v) - \sum_{(v,u)\in E} (f \uparrow f')(v,u)$$

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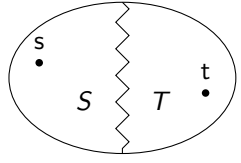
Cut, flow across and capacity of

Definition

A cut is a partition of V into subsets $S \ni s$ and $T \ni t$.



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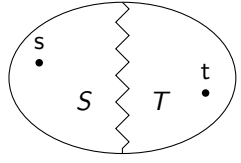

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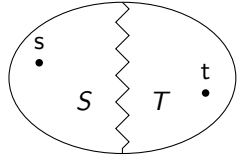

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Given a flow f in G, for all cuts (S, T) we have $f(S, T) = |f|$.

In other words, the net flow value across any cut (S, T) is equal to the value of the flow.

Proof.

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\begin{aligned}
f(S,T) &= \sum_{u \in S} \sum_{v \in T} (f(u,v) - f(v,u)) \\
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&= \sum_{u \in S} \sum_{v \in V} (f(u,v) - f(v,u)) \\
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Corollary (flow value upper bounded by cut capacity)

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For any flow f and any cut (S, T), $|f| \leq c(S, T)$.

Proof.

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\begin{aligned}
|f| &= f(S, T) && \text{(By Lemma 2)} \\
&= \sum_{u \in S} \sum_{v \in T} \big(f(u, v) - f(v, u)\big) && \text{(By definition of } f(S, T)\text{)} \\
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Illustrate on number line, all possible flow values are left of all possible cut capacities.

Max flow/Min cut Theorem says they meet in the middle.

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Max flow/Min cut Theorem

Theorem (Max flow/Min cut Theorem)

Let f be a flow in (G, s, t, c). Then the following 3 statements are equivalent:

1. *f is a max flow.*
2. *There is no augmenting path (in G_f).*
3. *There exists a cut (S, T) such that $|f| = c(S, T)$.*

Q: What does this say about Ford-Fulkerson?

Max flow/Min cut Theorem

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Q: What does this say about Ford-Fulkerson?
A: If it terminates, it returns the correct result.

Summary

Todays topic was Max Flow. We have covered

- Definition of flow network, flow, etc
- The Ford-Fulkerson Method
- The Max flow/Min cut Theorem

Next time:

- Proof of Max flow/Min cut Theorem
- Worst case analysis of Ford-Fulkerson
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Good Afternoon.

Advanced algorithms and data structures

Lecture 2: Max Flow 2

Jacob Holm (jaho@di.ku.dk)

November 20th 2024

Today's Lecture

Max flow
- Recap
- Ford-Fulkerson analysis
- Edmonds-Karp
- Integrality Theorem

Summary

Recap 1

Flow network (G, s, t, c), no self-loops or antiparallel edges.

Flow $f : V \times V \to \mathbb{R}$ satisfies:

1. $\forall u, v \in V : 0 \leq f(u, v) \leq c(u, v)$ (capacity constraints)
2. $\forall u \in V \setminus \{s, t\} : \sum_{v \in V} f(u, v) = \sum_{v \in V} f(v, u)$ (flow conservation)

Value $|f| = \sum_{v \in V} (f(s, v) - f(v, s))$.

Residual capacity $c_f : c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise} \end{cases}$

Residual network (G_f, s, t, c_f) where $G_f = (V, E_f)$ and $E_f = \{(u, v) \in V \times V \mid c_f(u, v) > 0\}$. This is a flow network.

Given flow f in G and f' in G_f,
$(f \uparrow f')(u, v) = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases}$

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1: **function** FORD-FULKERSON($G = (V, E)$, s, t, c)
2: $f \leftarrow 0$
3: **while** $\exists$ (augmenting) path p from s to t in G_f **do**
4: Find a max flow f_p along p in G_f.
5: $f \leftarrow f \uparrow f_p$
6: **return** f

A cut is a partition of V into sets $S \ni s$ and $T \ni t$.

Definition: $f(S, T) := \sum_{u \in S} \sum_{v \in T} f(u, v) - f(v, u)$.

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Lemma 1: $f \uparrow f'$ is a flow in G of value $|f \uparrow f'| = |f| + |f'|$.

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Assume for contradiction f is a max flow and there exists an augmenting path p. Then by Lemma 1, $f \uparrow f_p$ is a flow in G of value $|f \uparrow f_p| = |f| + |f_p| > |f|$, which is a contradiction. $\square$

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Let $S = \{v \in V \mid v \text{ is reachable from } s \text{ in } G_f\}$, $T = V \setminus S$. Then S, T partition V, $s \in S$ (why?) and $t \in T$ (why?), so (S, T) is a cut.
Now let $u \in S, v \in T$. Then $f(u, v) = c(u, v)$ (why?),

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&= c(S, T) && \text{(Definition of } c(S, T)) \qquad \square
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Max flow/Min cut Theorem

Given a flow f in G, the following 3 statements are equivalent:

1. f is a max flow.
2. There is no augmenting path (a path $s \rightsquigarrow t$ in G_f).
3. $\exists$ cut (S, T) such that $|f| = c(S, T)$.

(2) $\implies$ (3).

Let $S = \{v \in V \mid v \text{ is reachable from } s \text{ in } G_f\}$, $T = V \setminus S$. Then S, T partition V, $s \in S$ (why?) and $t \in T$ (why? no augmenting path), so (S, T) is a cut. Now let $u \in S, v \in T$. Then $f(u, v) = c(u, v)$ (why?), otherwise $c_f(u, v) > 0$ so $(u, v) \in E_f$. Since u is reachable from s in G_f that implies v is reachable from s in G_f thus $v \in S$ contradicting $v \in T = V \setminus S$.

Similarly, $f(v, u) = 0$ (why?),

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|f| &= f(S, T) && \text{(By Lemma 2)} \\
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Let (S, T) be the cut from (3), and f' be any other flow in G. By the Corollary, $|f'| \leq c(S, T) = |f|$, so f is a max flow. $\square$

Ford-Fulkerson worst case analysis

In general Ford-Fulkerson is not guaranteed to terminate,
i.e. there exists flow networks and bad choices of augmenting paths such that it never terminates.
For this to happen, some capacities need to be irrational.

If all capacities are integers, F.F. does at most $|f^*|$ iterations, where f^* is a max flow (why?).
Assuming each iteration can be done in $\mathcal{O}(E)$ time, that gives a running time of $\mathcal{O}(E \cdot |f^*|)$.

Bad case example:

Edmonds-Karp algorithm avoids the bad case by always choosing the shortest augmenting path.

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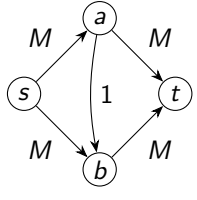

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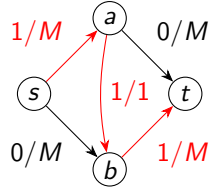
After iteration 1



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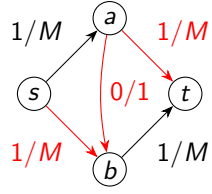
Bad case example:
After iteration 2



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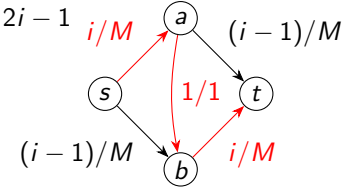
After iteration $2i-1$



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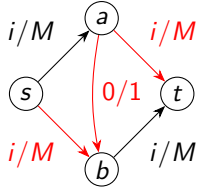

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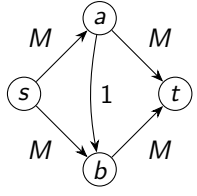

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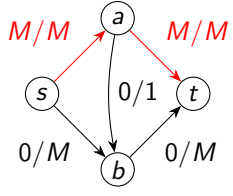

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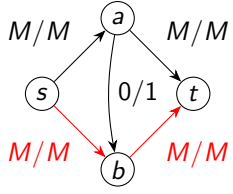

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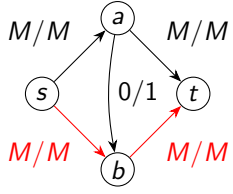

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Edmonds-Karp Algorithm

1: **function** EDMONDS-KARP($G = (V, E)$, s, t, c)
2: $f \leftarrow 0$
3: **while** $\exists$ (augmenting) path from s to t in G_f **do**
4: $p \leftarrow$ shortest such path.
5: Find a max flow f_p along p in G_f.
6: $f \leftarrow f \uparrow f_p$
7: **return** f

Theorem

The number of iterations of Edmonds-Karp is $\mathcal{O}(V \cdot E)$.

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Edmonds-Karp can be implemented to run in $\mathcal{O}(V \cdot E^2)$ time.

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Break

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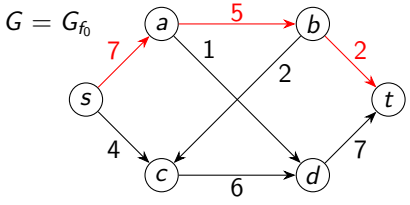

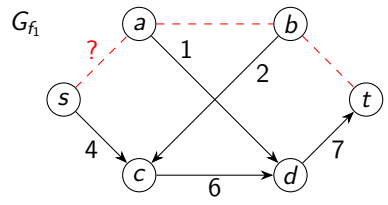

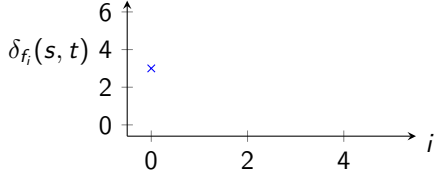

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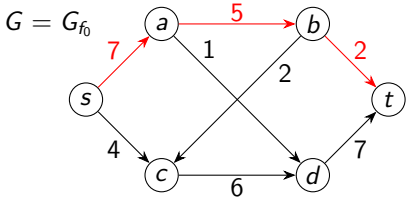

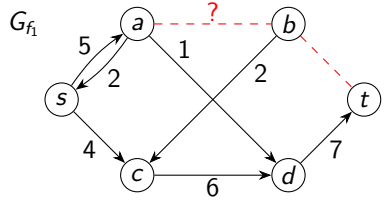

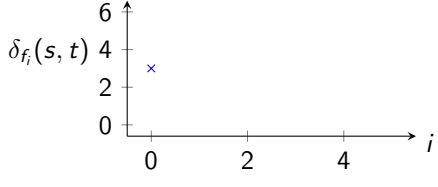

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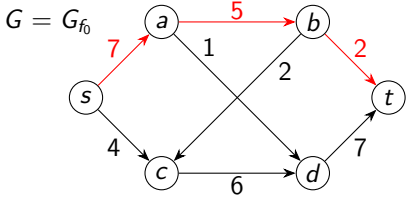

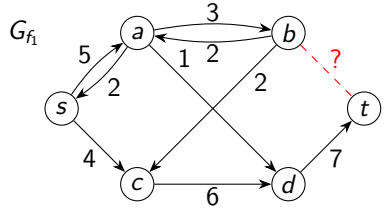

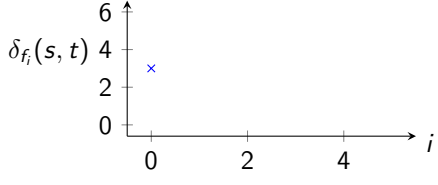

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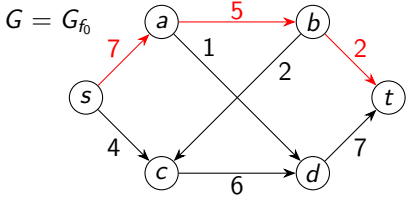
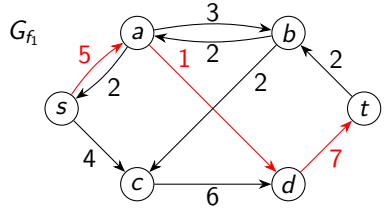
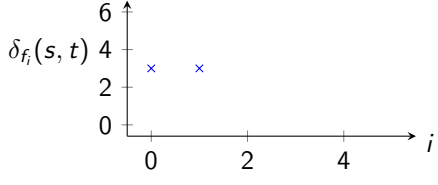
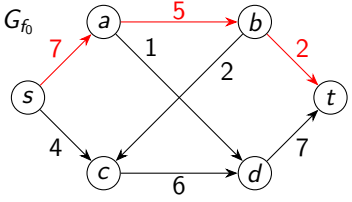




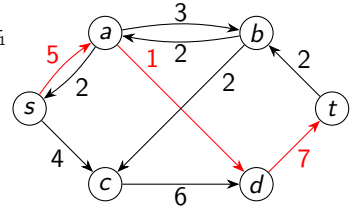


Edmonds-Karp Example

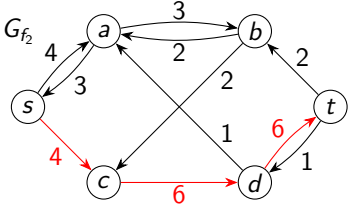
$G = G_{f_0}$

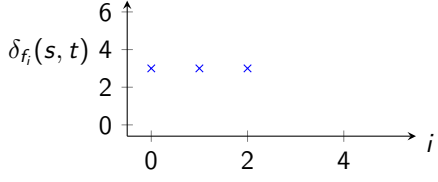


G_{f_1}



G_{f_2}





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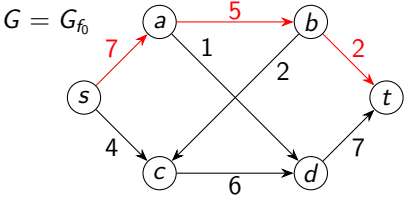
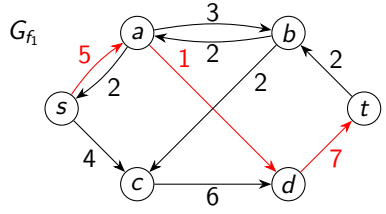
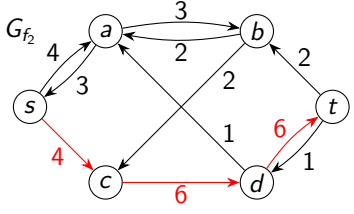
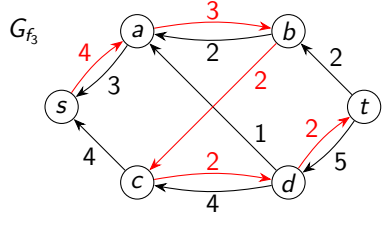
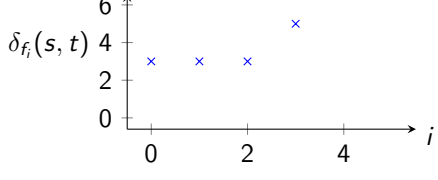


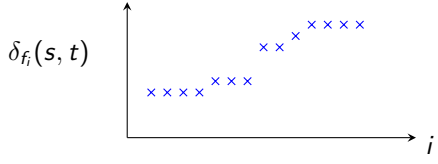








Edmonds-Karp Proof sketch of #iterations



We will show that the distance $\delta_{f_i}(s,t)$ is nondecreasing, and that it can only stay the same for at most E consecutive iterations.

This implies that #iterations in Edmonds-Karp is $\mathcal{O}(V \cdot E)$, why?

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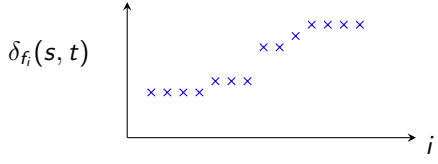

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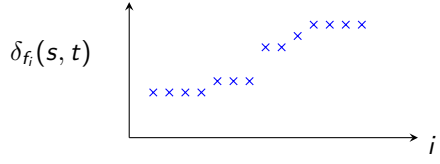

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So we have $\mathcal{O}(V)$ "runs", each of at most E consecutive iterations where the $s \rightsquigarrow t$ distance is unchanged. $\implies$ $\mathcal{O}(V \cdot E)$ iterations in total.

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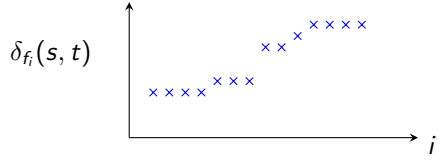

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Edmonds-Karp Level sets and forward/backward edges

Consider consecutive flows $f_0, \ldots, f_k$ found by Edmonds-Karp, where $\delta_{f_0}(s,t) = \delta_{f_1}(s,t) = \ldots = \delta_{f_k}(s,t)$.

For $d = 0, 1, \ldots$ let $L_d = \{v \in V \mid \delta_{f_0}(s,v) = d\}$, i.e. L_d is the "BFS layer" consisting of vertices at distance d from s in G_{f_0}.

A forward edge (of G_{f_i}) is an edge $(u,v) \in G_{f_i}$ such that for some d, $u \in L_d$ and $v \in L_{d+1}$.

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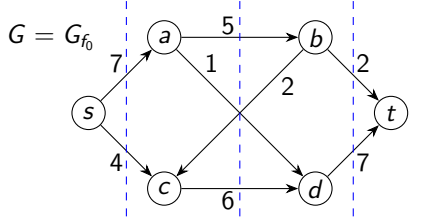

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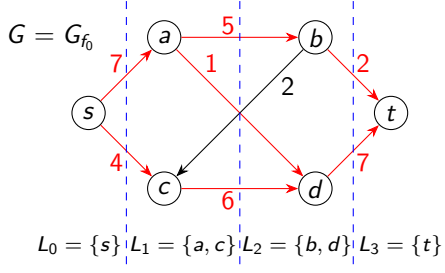

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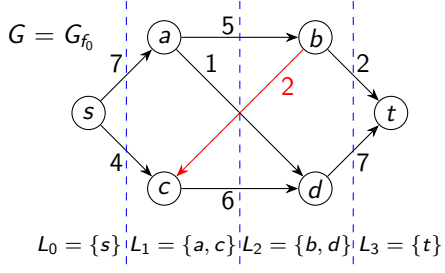

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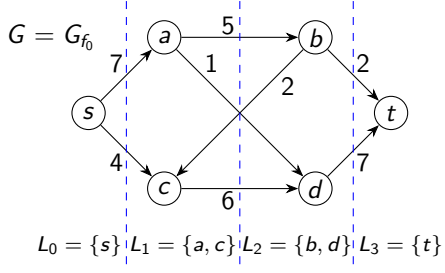
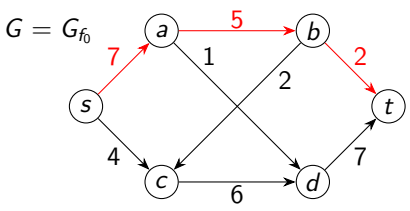
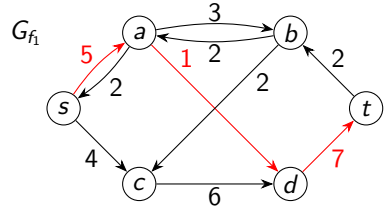
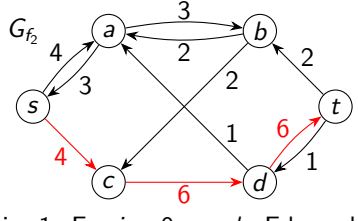


Edmonds-Karp Claim 1







Claim 1: For $i = 0, \ldots, k$, Edmonds-Karp finds an augmenting path in G_{f_i} consisting only of edges that are forward edges in G_{f_0}.

Note that Claim 1 implies $k = \mathcal{O}(E)$, why?

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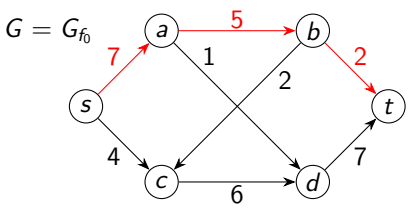

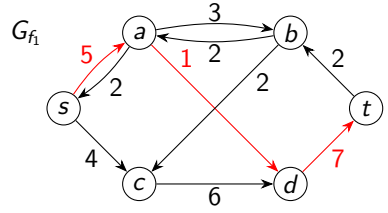

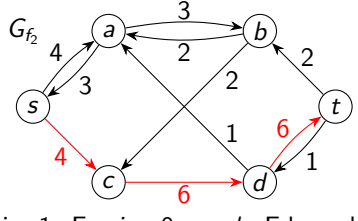

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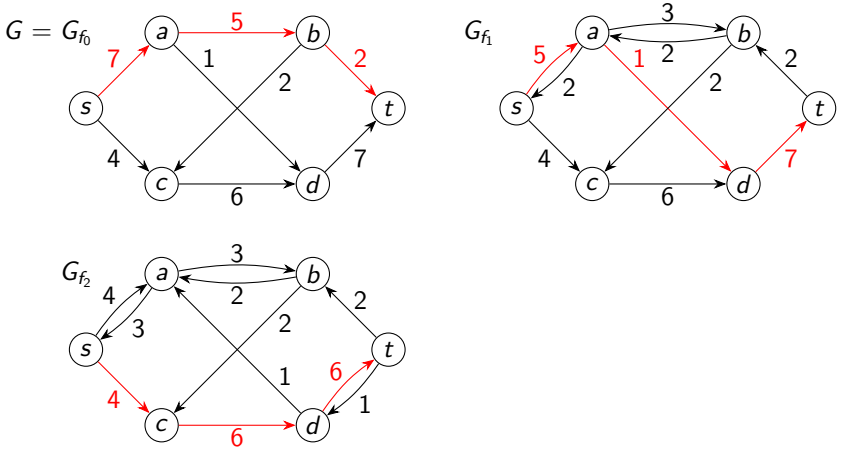

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At least one forward edge gets saturated in iteration i, which removes it from $G_{f_{i+1}}$.

Edmonds-Karp Claim 2

Claim 2: If there is an augmenting path in $G_{f_{k+1}}$, then $\delta_{f_{k+1}}(s,t) \geq \delta_{f_k}(s,t)$.

Note that Claim 1 and Claim 2 together gives the claimed #iterations for Edmonds-Karp.

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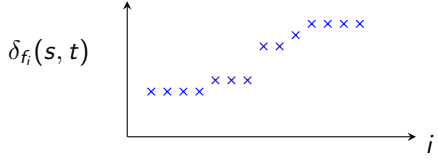

Edmonds-Karp Proof sketch for Claim 1 & 2

Claim 1: For $i = 0, \ldots, k$, Edmonds-Karp finds an augmenting path in G_{f_i} consisting only of edges that are forward edges in G_{f_0}.

Proof sketch.
This is clear for $i = 0$, because a shortest path can only use forward edges.
So suppose $1 \leq i \leq k$. G_{f_i} is obtained from $G_{f_{i-1}}$ by removing forward edges (at least one!) and adding backward edges.
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The claim follows by induction. □

Claim 2: If there is an augmenting path in $G_{f_{k+1}}$, then $\delta_{f_{k+1}}(s,t) \geq \delta_{f_k}(s,t)$.

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$G_{f_{k+1}}$ is obtained from G_{f_0} by removing forward edges and adding backward edges. This can never reduce the distance. □

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$G_{f_{k+1}}$ is obtained from G_{f_0} by removing forward edges and adding backward edges. This can never reduce the distance. □

Integrality Theorem

Integrality Theorem: Given integer capacities, Ford-Fulkerson (and therefore Edmonds-Karp) will find an integer-valued flow $f : V \times V \to \mathbb{Z}_{\geq 0}$ with $|f| \in \mathbb{Z}_{\geq 0}$ an integer.

Note that not all max flows in a network with integer capacities have to be integer-valued. Q: Can you find a max flow in this example that is not integer-valued?

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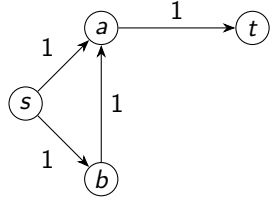

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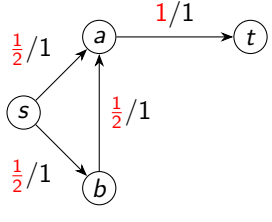

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AADS, Lecture 4

Randomized Algorithms

Jacob Holm (jaho@di.ku.dk)

November 27th 2024

Good afternoon. My name is Jacob Holm.
You can help by asking questions during class if there is anything that is not clear.
Remember, if it is not clear to you, then it is probably also unclear to at least one other person in the room.
You can help more than just yourself by asking for clarification.
I am also teaching the Randomized Algorithms course, and the next two lectures are a tiny taste of that.

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- Faster.
- Simpler code.
- Sometimes only option, e.g. Big Data, Streaming, Machine Learning, Security, (Differential) Privacy, etc.

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Therefore this course!

Todays Lecture

Quicksort
- Linearity of expectation
- Expectation of indicator variable

Min-Cut
- Conditional probabilities
- Time/error probability tradeoff

Las Vegas vs Monte Carlo

Summary

AADS Lecture 4 (RA), Part 1

Quicksort

Basic Quicksort [Hoare]

1: **function** $\mathrm{QS}(S = \{s_1, \ldots, s_n\})$
▷ Assumes all elements in S are distinct.
2: **if** $|S| \leq 1$ **then**
3: **return** list(S)
4: **else**
5: Pick pivot $x \in S$
6: $L \leftarrow \{y \in S \mid y < x\}$
7: $R \leftarrow \{y \in S \mid y > x\}$
8: **return** $\mathrm{QS}(L)+[x]+\mathrm{QS}(R)$

For each $y \in S \setminus \{x\}$, compare to y to x once

Q: Does anyone see what essential part is missing from this description?

Lemma

QS correctly sorts the numbers.

Proof.

By induction on n. $n = 0, 1$ is trivial, so assume it holds for up to $n-1$ numbers. Then by our induction hypothesis $\mathrm{QS}(L)$ and $\mathrm{QS}(R)$ are sorted, so $\mathrm{QS}(L)+[x]+\mathrm{QS}(R)$ is sorted. □

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Assuming we always pick the "middle" element we have $T(n) \leq 2T(n/2) + \mathcal{O}(n)$, so by the Master Theorem $T(n) \in \mathcal{O}(n \log n)$.

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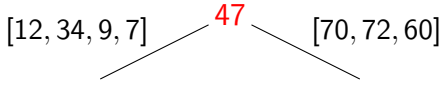

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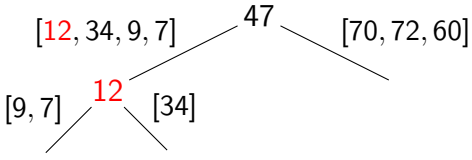

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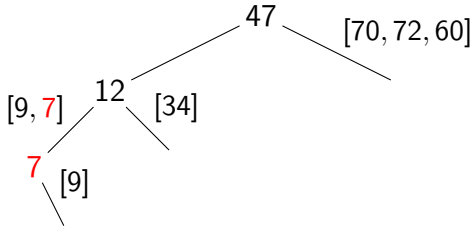

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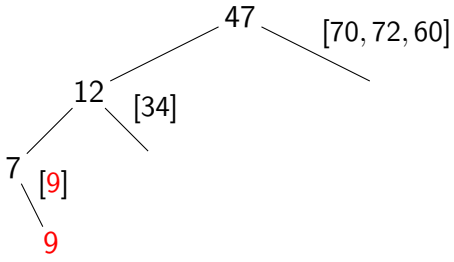

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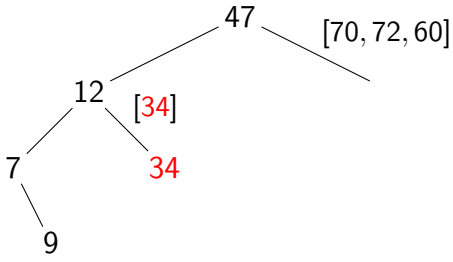

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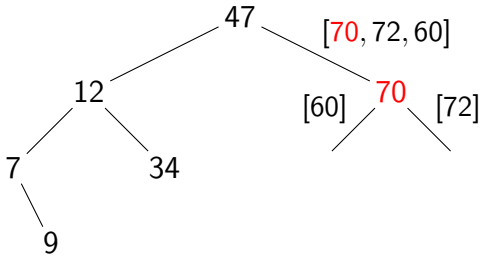

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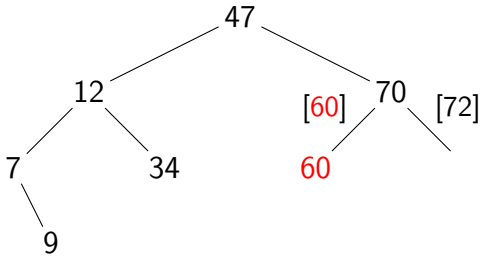

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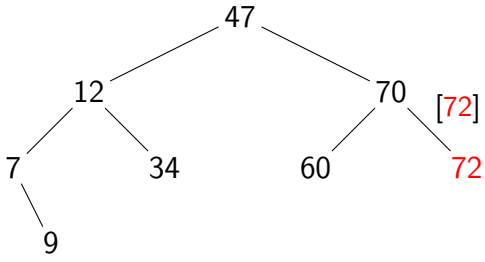

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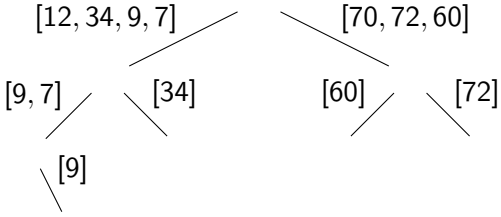

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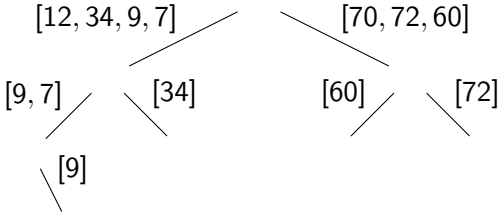

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$T(n) = T(0) + T(n-1) + \Theta(n)$, and thus $T(n) \in \Theta(n^2)$.

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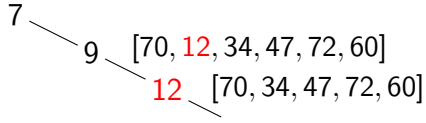

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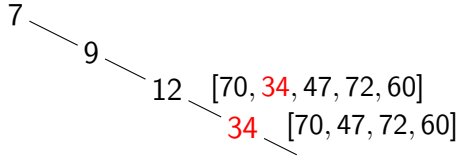

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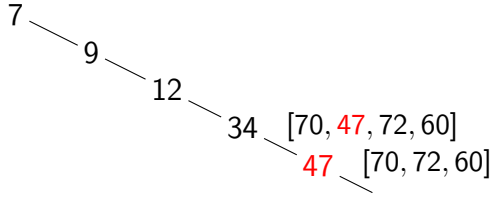

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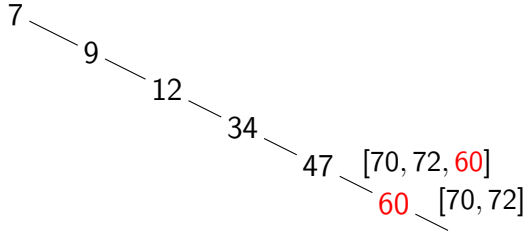

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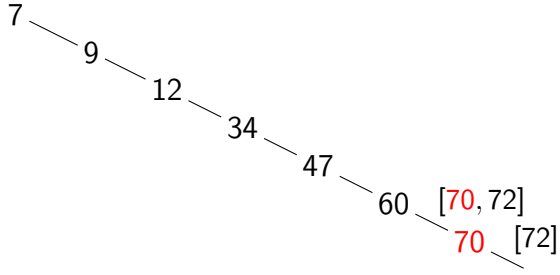

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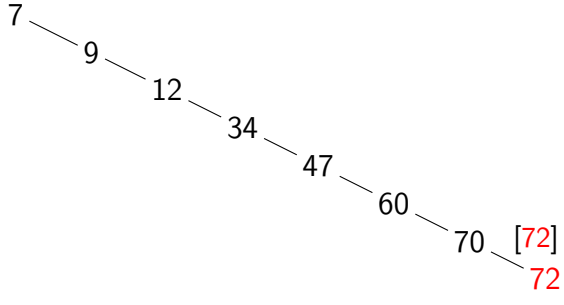

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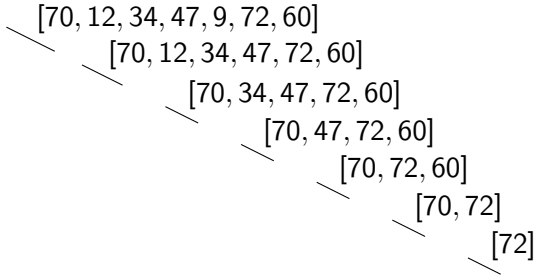

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Quicksort Example 2

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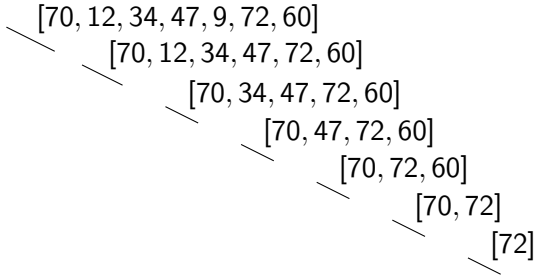

Total #comparisons: $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$

Assuming we always pick an "extreme" element we have $T(n) = T(0) + T(n-1) + \Theta(n)$, and thus $T(n) \in \Theta(n^2)$.

As you can see this looks like an extremely unbalanced binary search tree, so you might expect the number of comparisons to be large. Something like n^2.

Randomized Quicksort

1: **function** $\textsc{RandQS}(S = \{s_1, \ldots, s_n\})$
▷ Assumes all elements in S are distinct.
2: **if** $|S| \leq 1$ **then**
3: **return** S
4: **else**
5: Pick pivot $x \in S$, uniformly at random
6: $L \leftarrow \{y \in S \mid y < x\}$
7: $R \leftarrow \{y \in S \mid y > x\}$
8: **return** $\textsc{RandQS}(L)+[x]+\textsc{RandQS}(R)$

For each $y \in S \setminus \{x\}$, compare to y to x once

Randomized Quicksort, Analysis

Q: What is the expected number of comparisons?

Theorem
$\mathbb{E}[\#\textit{comparisons}] \in \mathcal{O}(n \log n)$

Let $[S_{(1)}, \ldots, S_{(n)}] := \textsc{RandQS}(S)$.
For $i < j$ let X_{ij} be the number of times that $S_{(i)}$ and $S_{(j)}$ are compared. We can then compute

$$\#\text{comparisons} = \sum_{i<j} X_{ij}$$

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Uses *linearity of expectation*:

$$\mathbb{E}[A + B] = \mathbb{E}[A] + \mathbb{E}[B]$$

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Randomized Quicksort, Analysis

Observe that $X_{ij} \in \{0, 1\}$ (why?).

Since $X_{ij} \in \{0, 1\}$, it is an *indicator variable* for the event that $S_{(i)}$ and $S_{(j)}$ are compared. Let p_{ij} be the probability of this event. Then

$$\begin{aligned}\mathbb{E}[X_{ij}] &= \sum_{x \in \{0,1\}} \Pr[X_{ij} = x] \cdot x \qquad \text{(Def. of } \mathbb{E}\text{)} \\ &= (1 - p_{ij}) \cdot 0 + p_{ij} \cdot 1 = p_{ij}\end{aligned}$$

Thus *the expectation of an indicator variable equals the probability of the indicated event.*

Therefore

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Lemma

$S_{(i)}$ and $S_{(j)}$ are compared iff $S_{(i)}$ or $S_{(j)}$ is first of $S_{(i)}, \ldots, S_{(j)}$ to be chosen as pivot.

Proof.

Each recursive call returns some sublist $[S_{(a)}, \ldots, S_{(b)}]$. Let $x = S_{(c)}$ be the pivot.

Suppose $a \leq i < j \leq b$. $\boxed{a}\boxed{\cdots}\boxed{i}\boxed{\cdots}\boxed{j}\boxed{\cdots}\boxed{b}$

$c < i$ or $c > j$: $S_{(i)}$ and $S_{(j)}$ not compared now, but together in recursion. Recursion stops when $i \leq c \leq j$.

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Randomized Quicksort, Analysis

Thus, p_{ij} is the conditional probability of picking $S_{(i)}$ or $S_{(j)}$ given that the pivot is picked uniformly at random in $\{S_{(i)}, S_{(i+1)}, \ldots, S_{(j)}\}$:

$$\begin{aligned} p_{ij} &= \Pr[c \in \{i,j\} \mid c \in \{i, i+1, \ldots, j\} \text{ u.a.r.}] \\ &= \frac{2}{|\{i, i+1, \ldots, j\}|} = \frac{2}{j+1-i} \end{aligned}$$

It follows that

$$\mathbb{E}[\#\text{comparisons}] = \sum_{i<j} p_{ij} = \sum_{i<j} \frac{2}{j+1-i}$$

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$$= \sum_{i=1}^{n-1} \sum_{k=2}^{n+1-i} \frac{2}{k} < \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{2}{k}$$

$$= 2n \sum_{k=2}^{n} \frac{1}{k} = 2n\left(\left(\sum_{k=1}^{n} \frac{1}{k}\right) - 1\right) = 2n(H_n - 1)$$

$$\leq 2n \int_1^n \frac{1}{x} dx = 2n \ln n \in \mathcal{O}(n \log n)$$

- From before.
- Expanding the $\sum_{i<j}$ notation.
- The denominator $k = j + 1 - i$ takes each value from $2, \ldots, n + 1 - i$ once.
- Since all terms are positive, adding more terms can only increase the value
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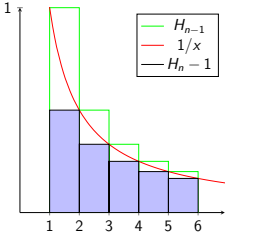

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Randomized Quicksort, Summary

When $|S| = n$, the expected number of comparisons done by $\textsc{RandQS}(S)$ is less than $2nH_n \in \mathcal{O}(n \log n)$ *for any input*.

Even stronger (see Problem 4.14), we can show that the number of comparisons is $\mathcal{O}(n \log n)$ *with high probability*.

AADS Lecture 4 (RA), Part 2

Min-Cut

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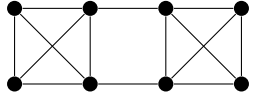
This is a related, but different notion of Min-Cut than in the "Max-Flow Min-Cut theorem" from lecture 1 and 2.

Problem: Given a connected graph $G = (V, E)$



Find smallest $C \subseteq E$ that splits G.

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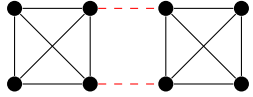

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Randomized Min-Cut [Karger & Stein]

```
1: function RandMinCut(V, E)
2:     while |V| > 2 and E ≠ ∅ do
3:         Pick e ∈ E uniformly at random.
4:         Contract e and remove self-loops.
5:     return E
```

Randomized Min-Cut, Example

1: **function** RandMinCut(V, E)
2: **while** $|V| > 2$ and $E \neq \emptyset$ **do**
3: Pick $e \in E$ uniformly at random.
4: Contract e and remove self-loops.
5: **return** E

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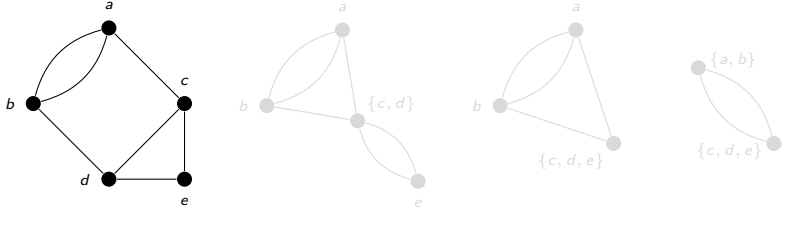

Randomized Min-Cut, Example

1: **function** RandMinCut(V, E)
2: **while** $|V| > 2$ and $E \neq \emptyset$ **do**
3: Pick $e \in E$ uniformly at random.
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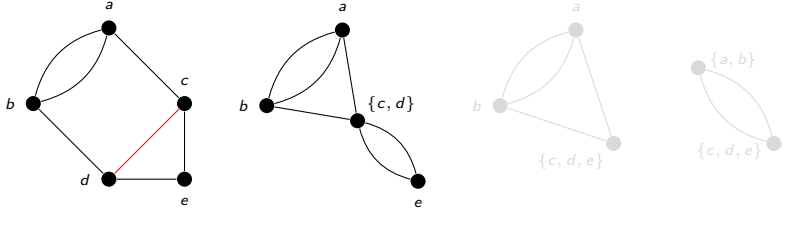

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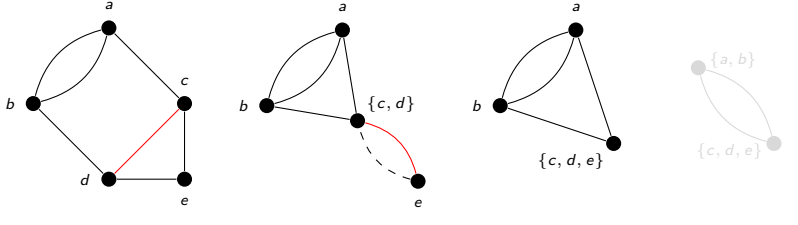

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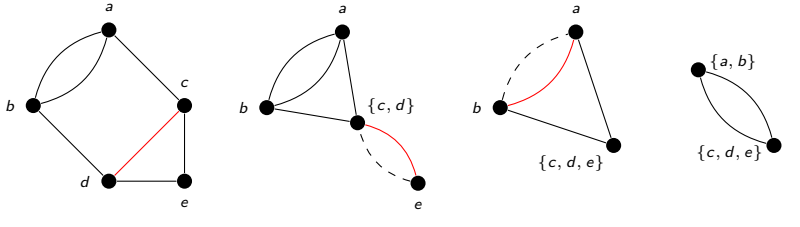

Randomized Min-Cut, Analysis

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1: function RandMinCut(V, E)
2:     while |V| > 2 and E ≠ ∅ do
3:         Pick e ∈ E uniformly at random.
4:         Contract e and remove self-loops.
5:     return E
```

Lemma

RandMinCut(G) *always returns a cut.*

Proof.

Proof by induction on the number k of iterations of the loop (note $k \leq n-2$). If $k = 0$ it is trivial, so suppose that it is true for up to $k-1$ iterations. The first iteration constructs graph G' by contracting an edge from G and removing self-loops, and then do at most $k-1$ futher iterations starting from G' so by the induction hypothesis we return a cut in G'. But every such cut is also a cut in G. □

Randomized Min-Cut, Analysis

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3:         Pick e ∈ E uniformly at random.
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Observation

RandMinCut(G) *may return a cut of size* $> \lambda(G)$.

Lemma

A specific min-cut C is returned iff no edge from C was contracted.

Randomized Min-Cut, Analysis

Theorem

For any min-cut C, the probability that $\textsc{RandMinCut}(G)$ *returns C is* $\geq \frac{2}{n(n-1)}$.

Let $e_1, \ldots, e_{n-2}$ be the contracted edges, let $G_0 = G$ and $G_i = G_{i-1}/e_i$.
Let $\mathcal{E}_i$ be the (good) event that $e_i \notin C$.
C is returned iff $\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{n-2}$.

Goal: $\Pr[\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{n-2}] \geq \frac{2}{n(n-1)}$

- In words, $\mathcal{E}_i$ is the event that the ith edge contracted is not in C, i.e., the ith contraction does not destroy C.
- $\mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{n-2}$ is thus the event that C is not destroyed in any step of the algorithm.

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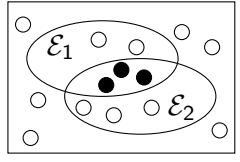
Conditional Probabilities

Given events $\mathcal{E}_1, \mathcal{E}_2$ with $\Pr[\mathcal{E}_1] > 0$, the *conditional probability* of $\mathcal{E}_2$ given $\mathcal{E}_1$ is defined as

$$\Pr[\mathcal{E}_2|\mathcal{E}_1] = \frac{\Pr[\mathcal{E}_1 \cap \mathcal{E}_2]}{\Pr[\mathcal{E}_1]}$$



It follows that

$$\Pr[\mathcal{E}_1 \cap \mathcal{E}_2] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2|\mathcal{E}_1]$$

And in general for events $\mathcal{E}_1, \ldots, \mathcal{E}_k$

$$\Pr[\cap_{i=1}^{k} \mathcal{E}_i] = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2|\mathcal{E}_1] \cdots \Pr[\mathcal{E}_k| \cap_{i=1}^{k-1} \mathcal{E}_i]$$

This is easy to prove by induction.

Randomized Min-Cut, Proof

$$
\begin{aligned}
&\Pr[\text{specific min-cut } C \text{ returned}] \\
&\quad = \Pr[\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{n-2}] \\
&\quad = \Pr[\mathcal{E}_1] \cdot \Pr[\mathcal{E}_2 | \mathcal{E}_1] \cdots \Pr[\mathcal{E}_{n-2} | \mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{n-3}] \\
&\quad = \prod_{i=1}^{n-2} p_i \quad \text{where } p_i = \Pr[\mathcal{E}_i | \mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{i-1}]
\end{aligned}
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Randomized Min-Cut, Proof

$G_i = (V_i, E_i)$ has $n_i = n - i$ vertices. (why?)
Contractions can not decrease the min-cut size
so $\lambda(G_i) \geq |C|$.
It follows that each vertex v of G_i has degree $d_i(v)$ at least $|C|$.
Summing up all degrees of G_i,

$$|E_i| = \frac{1}{2} \sum_{v \in V_i} d_i(v) \geq \frac{1}{2} n_i |C|.$$

- Each contraction reduces the number of vertices by 1.
- Every cut in G_i is a cut in G_{i-1} and therefore in G.
- Note that the edges incident to a vertex v form a cut and so $d_i(v) \geq \lambda(G_i) \geq |C|$.
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Randomized Min-Cut, Proof

We have shown that $G_i = (V_i, E_i)$ has $n_i = n - i$ vertices and at least $|E_i| \geq \frac{1}{2} n_i |C|$ edges. We want to lower bound

$$p_i = \Pr[\text{uniformly random } e \in E_{i-1} \text{ is not in } C \mid \cap_{j=1}^{i-1} \mathcal{E}_j]$$

The complementary probability, i.e. the probability of picking an edge of C in the ith iteration, given that no edge of C has been picked in a previous iteration, is

$$\begin{aligned} 1 - p_i &= \Pr[\text{uniformly random } e \in E_{i-1} \text{ is in } C \mid \cap_{j=1}^{i-1} \mathcal{E}_j] \\ &= \frac{|C|}{|E_{i-1}|} \leq \frac{|C|}{\frac{1}{2} n_{i-1} |C|} = \frac{2}{n_{i-1}} = \frac{2}{n - (i-1)} \\ \Rightarrow p_i &\geq 1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1} \end{aligned}$$

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so now we can lower bound p_i.

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&\geq \prod_{i=1}^{n-2} \frac{n-1-i}{n+1-i} \\
&= \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \\
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Telescoping product.

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Randomized Min-Cut, Summary

So for given min-cut C, $\Pr[C \text{ is returned}] \geq \frac{2}{n(n-1)}$.

Is this tight? I.e. do we have examples matching this bound?

Is this probability good?

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Is this probability good?

How can we improve it?

Randomized Min-Cut, Tradeoff

Imagine calling $\textsc{RandMinCut}(G)$ $t\frac{n(n-1)}{2}$ times and letting $C^\star$ be the smallest cut returned.

$$\Pr[C^\star \text{ is not a min-cut}] \leq \left(1 - \frac{2}{n(n-1)}\right)^{t\frac{n(n-1)}{2}}$$

$$\leq \left(e^{-\frac{2}{n(n-1)}}\right)^{t\frac{n(n-1)}{2}}$$

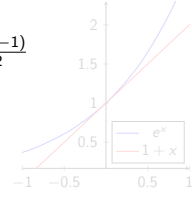
(This uses that $1 + x \leq e^x$ for all $x \in \mathbb{R}$, see Proposition B.3.1)

$$= e^{-t}$$

$$\Pr[C^\star \text{ is a min-cut}] \geq 1 - e^{-t}$$

$$= 1 - n^{-c} \qquad \text{(If we set } t = c \ln n\text{)}$$

Thus for any $c > 0$ if we repeat $c \cdot \frac{n(n-1)}{2} \cdot \ln n$ times, the probability of getting a min-cut is at least $1 - n^{-c}$. We call this *high probability of success*.



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- Since the calls to $\textsc{RandMinCut}(G)$ are independent, the probability that no min-cut is among the cuts returned is the product.
- $1 + x \leq e^x$
- Choosing e.g. $t = 21$ we reduce the error probability to around one in a billion.
- Choosing $t = c \ln n$ for constant c, we get a *high probability of success*, namely at least $1 - e^{-c \ln n} = 1 - 1/n^c$.
- We thus get a tradeoff between running time and probability of success.

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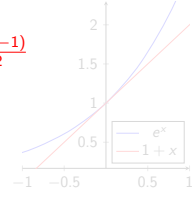

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- Choosing e.g. $t = 21$ we reduce the error probability to around one in a billion.
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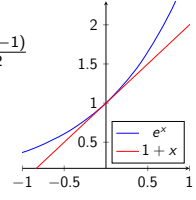
- We thus get a tradeoff between running time and probability of success.

Randomized Min-Cut, Tradeoff

Imagine calling $\textsc{RandMinCut}(G)$ $t\frac{n(n-1)}{2}$ times and letting $C^\star$ be the smallest cut returned.

$$\Pr[C^\star \text{ is not a min-cut}] \leq \left(1 - \frac{2}{n(n-1)}\right)^{t\frac{n(n-1)}{2}}$$

$$\leq \left(e^{-\frac{2}{n(n-1)}}\right)^{t\frac{n(n-1)}{2}}$$



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$$\Pr[C^\star \text{ is a min-cut}] \geq 1 - e^{-t}$$

$$= 1 - n^{-c} \qquad \text{(If we set } t = c \ln n\text{)}$$

Thus for any $c > 0$ if we repeat $c \cdot \frac{n(n-1)}{2} \cdot \ln n$ times, the probability of getting a min-cut is at least $1 - n^{-c}$. We call this *high probability of success*.

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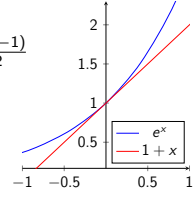

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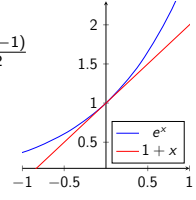

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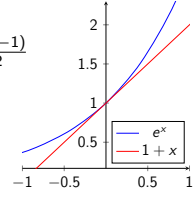

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Randomized Min-Cut, Simple implementation

In practice, using a "Union-Find" data structure.

```
1:  function RandMinCut(V, E)
2:      for u ∈ V do
3:          MAKE-SET(u)
4:      C ← ∅, π ← a random permutation of E, r ← |V|
5:      for uv ∈ E in the order π do
6:          p_u ← FIND(u), p_v ← FIND(v)
7:          if p_u ≠ p_v then
8:              if r > 2 then
9:                  r ← r − 1
10:                 UNION(p_u, p_v)
11:             else
12:                 C ← C ∪ {uv}
13:     return C
```

The running time for this is $\mathcal{O}(m\alpha(n))$. Running it $\mathcal{O}(n^2 \log n)$ times to get high probability takes $\mathcal{O}(n^2 m\alpha(n) \log n)$ time.

Deterministic Min-Cut

Pick arbitrary $s \in V$. For each $t \in V \setminus \{s\}$ compute max-flow from s to t. Return the minimum.

What is the running time?

Best algorithm known (2024): $\widetilde{\mathcal{O}}(m)$-time algorithm by Kawarabayashi and Thorup [JACM 2019].

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Deterministic Min-Cut

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What is the running time? We run Ford-Fulkerson $n-1$ times. Each run takes $\mathcal{O}(m|f^*|)$ time where $|f^*| \leq m$. In total $\mathcal{O}(nm^2)$ time. For dense graphs this is much worse than $\mathcal{O}(n^2 m\alpha(n)\log n)$.

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Edmonds-Karp does not help us here, at least not without a better analysis. For unit-capacity graphs like this one, the bound $\mathcal{O}(m|f^*|) = O(m^2)$ is better than the $\mathcal{O}(nm^2)$ we get for Edmonds-Karp. On the other hand, since Edmonds-Karp *is* a version of Ford-Fulkerson, you can/should still use it, as the same improved analysis applies.

Las Vegas vs Monte Carlo

What is the main difference between the guarantees of RandQS and RandMinCut?

Las Vegas vs Monte Carlo

What is the main difference between the guarantees of RandQS and RandMinCut?

Las Vegas: Always returns correct answer. #steps used is a random variable.

Monte Carlo: Some probability of error. #steps used may be random or not.

Converting L.V. $\leftrightarrow$ M.C.

As part of Assignment 2 you will prove that we can sometimes convert a Monte Carlo algorithm into a Las Vegas algorithm.

How about the other direction? Can we always take a Las Vegas algorithm running in expected $\mathcal{O}(f(n))$ time and turn it into a Monte Carlo algorithm running in worst case $\mathcal{O}(f(n))$ time?

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Next time

Hashing.

Good Afternoon.

Advanced algorithms and data structures

Lecture 5: Hashing

Jacob Holm (jaho@di.ku.dk)

December 2nd 2024

Today's Lecture

Hashing

- Hashing fundamentals
- Application: Unordered sets/Hashing with chaining
- Application: Signatures
- Practical hash functions
- Application: Coordinated sampling

Preliminaries

Notation:
For $n \in \mathbb{N}$:

$$[n] = \{0, \dots, n-1\}$$
$$[n]_+ = \{1, \dots, n-1\}$$

Iverson bracket:

$$[\text{condition}] = \begin{cases} 1 & \text{if condition is true} \\ 0 & \text{if condition is false} \end{cases}$$

For a random variable X:

$$\mu_X = \mathbb{E}[X] \quad \text{(expectation)}$$
$$\text{Var}[X] = \mathbb{E}[(X - \mu_X)^2] \quad \text{(variance)}$$
$$\sigma_X = \sqrt{\text{Var}[X]} \quad \text{(std. deviation)}$$

Inequalities:
Expectation of indicator variable X:

$$\mathbb{E}[X] = \Pr[X = 1]$$

Linearity of expectation:

$$\mathbb{E}\Big[\sum_i X_i\Big] = \sum_i \mathbb{E}[X_i]$$

Sum of pairwise indep. variances:

$$\text{Var}\Big[\sum_i X_i\Big] = \sum_i \text{Var}[X_i]$$

Union bound:

$$\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$$

Markovs Inequality: For $X \geq 0, t > 0$

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t} = \frac{\mu_X}{t}$$

Chebyshevs Inequality: For $t > 0$

$$\Pr\Big[|X - \mu_X| \geq t\sigma_X\Big] \leq \frac{1}{t^2}$$

AADS Lecture 5 (Hashing), Part 1

Hashing fundamentals

Hash function

Given a (typically large) universe U of keys, and a positive integer m.

Definition

A *random hash function* $h : U \to [m]$ is a randomly chosen function from some family of functions mapping U to $[m]$. Equivalently, it is a function h such that for each $x \in U$, $h(x) \in [m]$ is a random variable.

Cryptographic "hash functions" such as MD5, SHA-1, and SHA-256 are not *random* hash functions, and do not have most of the properties we want here. Do not confuse them!

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When discussing random hash functions, we usually care about

1. Space (*seed size*) needed to represent h.
2. Time needed to calculate $h(x)$ given $x \in U$.
3. Properties of the random variable.

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Definition

A hash function $h : U \to [m]$ is *truly random* if the variables $h(x)$ for $x \in U$ are independent and uniform in $[m]$.

Impractical, why?

Definition

A random hash function $h : U \to [m]$ is *universal* if, for all $x \neq y \in U$: $\Pr_h[h(x) = h(y)] \leq \frac{1}{m}$.

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A random hash function $h : U \to [m]$ is *strongly universal* (a.k.a. 2-independent) if,

- Each key is hashed uniformly into $[m]$.
  (i.e. $\forall x \in U, q \in [m] : \Pr_h[h(x) = q] = \frac{1}{m}$)
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There are $m^{|U|}$ possible functions from U to $[m]$, so it takes at least $\log_2(m^{|U|}) = |U| \log_2 m$ bits to store which one we picked.

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We use $\Pr_h[\cdots]$ or $\Pr_h[\cdots]$ instead of just $\Pr[\cdots]$ to make it clear that the probability is based on the random choice of h, and *not* on e.g. x, y being chosen at random.

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For many purposes c-approximately universal hash functions for some small constant c are enough. We will see examples of such functions a little later today.

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Implying that for all $x \neq y \in U$, and $q, r \in [m]$:
$\Pr_h[h(x) = q \wedge h(y) = r] \leq ?$ (See Assignment 3 exercise 3.2).

AADS Lecture 5 (Hashing), Part 2

Application: Unordered sets/Hashing with chaining

Unordered sets/Dictionaries

Maintain a set S of at most n keys from some unordered universe U, under

$\textsc{insert}(x, S)$ Insert key x into S.

$\textsc{delete}(x, S)$ Delete key x from S.

$\textsc{member}(x, S)$ Return $x \in S$.

We could use some form of balanced tree to store S, but they usually take $\mathcal{O}(\log n)$ or $\mathcal{O}(\log \log U)$ time per operation, and we want each operation to run in expected constant time.

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Unordered sets/Dictionaries

Maintain a set S of at most n keys from some unordered universe U, under

INSERT(x, S) Insert key x into S.

DELETE(x, S) Delete key x from S.

MEMBER(x, S) Return $x \in S$.

We could use some form of balanced tree to store S, but they usually take $\mathcal{O}(\log n)$ or $\mathcal{O}(\log \log U)$ time per operation, and we want each operation to run in expected constant time.

Hashing with chaining

Idea: Pick $m \geq n$ and a *universal* $h : U \to [m]$.
Store array L, where
$L[i] =$ linked list over $\{y \in S \mid h(y) = i\}$.

Then $x \in S \iff x \in L[h(x)]$.

Each operation takes $\mathcal{O}(|L[h(x)]| + 1)$ time.

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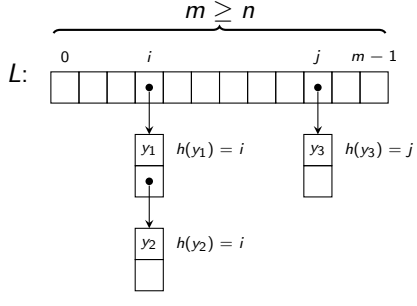

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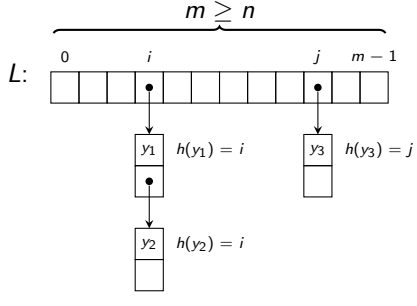

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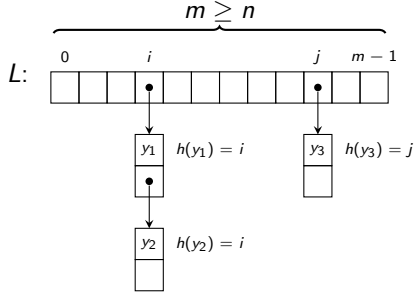

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Each operation takes $\mathcal{O}(|L[h(x)]| + 1)$ time.

Hashing with chaining

Theorem
For $x \notin S$, $\mathbb{E}_h\Big[\big|L[h(x)]\big|\Big] \le \frac{n}{m}$ $\qquad \Big(\le 1\Big)$

Proof.

$$\begin{aligned}
\mathbb{E}_h\Big[\big|L[h(x)]\big|\Big] &= \mathbb{E}_h\Big[\big|\{y \in S \mid h(y) = h(x)\}\big|\Big] \\
&= \mathbb{E}_h\Big[\sum_{y \in S}[h(y) = h(x)]\Big] \\
&= \sum_{y \in S}\mathbb{E}_h\Big[[h(y) = h(x)]\Big] \\
&= \sum_{y \in S}\Pr_h[h(y) = h(x)] \\
&\le |S|\tfrac{1}{m} \le \tfrac{n}{m} \le 1
\end{aligned}$$

□

Hashing with chaining

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$$\mathbb{E}_h\Big[\big|L[h(x)]\big|\Big] = \mathbb{E}_h\Big[\big|\{y \in S \mid h(y) = h(x)\}\big|\Big]$$

By definition of $L[i] := \{y \in S \mid h(y) = i\}$.

$$= \mathbb{E}_h\Big[\sum_{y \in S} [h(y) = h(x)]\Big]$$

$$= \sum_{y \in S} \mathbb{E}_h\Big[[h(y) = h(x)]\Big]$$

$$= \sum_{y \in S} \Pr_h[h(y) = h(x)]$$

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&\leq |S|\tfrac{1}{m} \leq \tfrac{n}{m} \leq 1
\end{aligned}$$

□

Here we use the *Iverson Bracket* notation

$$[P] := \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false} \end{cases}$$

This can often be used as a shorthand for an indicator variable.
In this case $[h(y) = h(x)]$ becomes an indicator variable for the event $h(y) = h(x)$.

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Linearity of expectation.

Hashing with chaining

Expectation of indicator variable

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For $x \notin S$, $\mathop{\mathbb{E}}_{h}\Big[\big|L[h(x)]\big|\Big] \leq \frac{n}{m} \qquad \Big(\leq 1\Big)$

Proof.

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Hashing with chaining

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\end{aligned}$$

□

Since $x \notin S$ and $y \in S$, we have $x \neq y$.
Then by definition of a universal hash function $h : U \to [m]$, $\Pr_h[h(y) = h(x)] \leq \frac{1}{m}$.

Hashing with chaining

Similarly, one can show that:

Theorem (Exercise)

For $x \in S$, $\mathop{\mathbb{E}}_{h}\Big[\big|L[h(x)]\big|\Big] \leq 1 + \frac{n-1}{m} \qquad \Big(\leq 2\Big)$

So we can conclude that each of the operations $\textsc{insert}(x, S)$, $\textsc{delete}(x, S)$, and $\textsc{member}(x, S)$ takes expected constant time when using hashing with chaining with a universal hash function.

Hashing with chaining

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AADS Lecture 5 (Hashing), Part 3

Application: Signatures

Application: Signatures

Problem: Assign a unique "signature" to each $x \in S \subseteq U$, $|S| = n$.

Solution: Use universal hash function $s : U \to [n^3]$.

Then we fail (by having a collision) with probability

$$\begin{aligned}
&\Pr_s[\exists \{x, y\} \subseteq S \text{ with } s(x) = s(y)] \\
&\quad = \Pr_s\Big[\bigcup_{\{x,y\}\subseteq S} \text{event } s(x) = s(y)\Big] \\
&\quad \leq \sum_{\{x,y\}\subseteq S} \Pr_s[s(x) = s(y)] \qquad \text{(Union bound)} \\
&\quad \leq \frac{\binom{n}{2}}{n^3} \qquad (s \text{ universal}) \\
&\quad < \frac{1}{2n}
\end{aligned}$$

Thus with "high probability" we have no collisions.

It is tempting to try to calculate a probability of no collisions more directly as a product over all pairs of the probability of that pair not colliding, i.e. as:

$$\begin{aligned}
\prod_{\{x,y\}\in S} \Pr_s[s(x) \neq s(y)] &\geq \left(1 - \frac{1}{n^3}\right)^{\binom{n}{2}} \\
&= \left(1 + \frac{1}{n^3 - 1}\right)^{-\binom{n}{2}} \\
&\geq e^{\left(\frac{-\binom{n}{2}}{n^3-1}\right)} \\
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While this happens to give the right answer in this case, the calculation is invalid because the events are not independent.

We use the union bound instead because that does not require independence.

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AADS Lecture 5 (Hashing), Part 4

Practical hash functions

Multiply-mod-prime

Let $U = [u]$ and pick prime $p \geq u$. For any $a, b \in [p]$, and $m < u$, let $h^m_{a,b} : U \to [m]$ be

$$h^m_{a,b}(x) = ((ax + b) \bmod p) \bmod m$$

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Is this a random hash function?

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Is this a random hash function? NO!

Multiply-mod-prime

Let $U = [u]$ and pick prime $p \geq u$. For any $a, b \in [p]$, and $m < u$, let $h^m_{a,b} : U \to [m]$ be

$$h^m_{a,b}(x) = ((ax + b) \bmod p) \bmod m$$

Choose $a, b \in [p]$ independently and uniformly at random, and let $h(x) := h^m_{a,b}(x)$.
Then $h : U \to [m]$ is a 2-approximately strongly universal hash function.

Multiply-shift

Let $U = [2^w]$ and $m = 2^\ell$. For any odd $a \in [2^w]$ define

$$h_a(x) := \left\lfloor \frac{(ax) \bmod 2^w}{2^{w-\ell}} \right\rfloor$$

Choose odd $a \in [2^w]$ uniformly at random, and let $h(x) := h_a(x)$.
Then $h : U \to [m]$ is a 2-approximately universal hash function.

(Assignment 3 exercise 3.4 asks you to show that it is not c-approximately strongly universal for any constant c).

Multiply-shift, C

For $U = [2^{64}]$ the C code looks like this:

```
#include<stdint.h>
uint64_t hash(uint64_t x, uint64_t l, uint64_t a)
{
  return (a*x) >> (64-l);
}
```

Reminder: Strongly universal

Recall:

Definition

A random hash function $h : U \to [m]$ is *c-approximately strongly universal* (a.k.a. 2-independent) if,

- Each key is hashed c-approximately uniformly into $[m]$.
  (i.e. $\forall x \in U, q \in [m] : \Pr_h[h(x) = q] = \frac{1}{m}$)
- Any two distinct keys hash independently.

Strong Multiply-shift

Let $U = [2^w]$ and $m = 2^\ell$, and pick $\bar{w} \geq w + \ell - 1$. For any pair $(a, b) \in [2^{\bar{w}}]^2$ define

$$h_{a,b}(x) := \left\lfloor \frac{(ax + b) \bmod 2^{\bar{w}}}{2^{\bar{w}-\ell}} \right\rfloor$$

Choose $a, b \in [2^{\bar{w}}]$ independently and uniformly at random, and let $h(x) := h_{a,b}(x)$.
Then $h : U \to [m]$ is a strongly universal hash function.

Strong Multiply-shift, C

For $\ell \leq w = 32$ and $\bar{w} = 64$ we have $U = [2^{32}]$ and the C code looks like this:

```
#include<stdint.h>
uint32_t hash(uint32_t x, uint32_t l,
              uint64_t a, uint64_t b)
{
  return (a*x+b) >> (64-l);
}
```

AADS Lecture 5 (Hashing), Part 5

Application: Coordinated sampling

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Suppose we have a bunch of *agents* that each observe some set of events from some universe U. Let $A_i \subseteq U$ denote the set of events seen by agent i, and suppose $|A_i|$ is large so only a small sample $S_i \subseteq A_i$ is actually stored.

At some later point we are interested in a subset $A^\star \subseteq \bigcup_i A_i$ (not known in advance), and in particular we want to estimate the size $|A^\star|$.

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Application: Coordinated sampling

If each agent independently just samples a random subset of the seen events, there is very little chance that two agents that see an event make the same decision.

$\Longrightarrow$ The samples are incomparable (almost useless).

Coordinated sampling means that all agents that see an event make the same decision about whether to store it.

$\Longrightarrow$ Samples can be combined, i.e.

- $S_i \cup S_j$ is a sample of $A_i \cup A_j$
- $S_i \cap S_j$ is a sample of $A_i \cap A_j$

In particular, $A^\star \cap \bigcup_i S_i$ is a sample of $A^\star$ and can be computed from $\bigcup_i S_i$ just by determining for each element whether it belongs to $A^\star$.

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Application: Coordinated sampling

Let $h : U \to [m]$ be a strongly universal hash function, and let $t \in \{0, \ldots, m\}$. Send h and t to all the agents.

Each agent samples $x \in U$ iff $h(x) < t$.

Thus if an agent sees the set $A_i \subseteq U$, the set $S_{h,t}(A_i) := \{x \in A_i \mid h(x) < t\}$ is sampled. Note that

- $S_{h,t}(A_i) \cup S_{h,t}(A_j) = S_{h,t}(A_i \cup A_j)$
- $S_{h,t}(A_i) \cap S_{h,t}(A_j) = S_{h,t}(A_i \cap A_j)$

Each $x \in U$ (if seen) is sampled with probability $\Pr_h[h(x) < t] = \frac{t}{m}$. Why?

For any $A \subseteq U$, $\mathbb{E}_h[|S_{h,t}(A)|] = |A| \cdot \frac{t}{m}$.

Thus we have an unbiased estimate $|A^\star| \approx \frac{m}{t} \cdot |S_{h,t}(A^\star)| = \frac{m}{t} \cdot |A^\star \cap \bigcup_i S_i|$.

How good is this estimate, i.e. what can we say about the *relative error* $:= \left|\frac{\text{estimated value} - \text{actual value}}{\text{actual value}}\right| = \left|\frac{\text{estimated value}}{\text{actual value}} - 1\right|$?

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$$\begin{aligned}
\mathop{\mathbb{E}}_h\big[|S_{h,t}(A)|\big] &= \mathop{\mathbb{E}}_h\left[\sum_{x \in A} [h(x) < t]\right] \\
&= \sum_{x \in A} \mathop{\mathbb{E}}_h\big[[h(x) < t]\big] \\
&= \sum_{x \in A} \Pr_h[h(x) < t] \\
&= \sum_{x \in A} \tfrac{t}{m} \\
&= |A| \cdot \tfrac{t}{m}
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Lemma

Let $X = \sum_{a \in A} X_a$ where the X_a are pairwise independent 0–1 variables. Let $\mu = \mathbb{E}[X]$. Then $\mathrm{Var}[X] \leq \mu$, and for any $q > 0$,

$$\Pr[|X - \mu| \geq q\sqrt{\mu}] \leq \tfrac{1}{q^2}$$

Proof (not curriculum).

For $a \in A$ let $p_a = \Pr[X_a = 1]$. Then $p_a = \mathbb{E}[X_a]$ and

$$\begin{aligned}\mathrm{Var}[X_a] &= \mathbb{E}[(X_a - p_a)^2] = (1 - p_a)(0 - p_a)^2 + p_a(1 - p_a)^2 \\ &= (p_a^2 + p_a(1 - p_a))(1 - p_a) = p_a(1 - p_a) \leq p_a \\ \mathrm{Var}[X] &= \mathrm{Var}\Big[\sum_{a \in A} X_a\Big] = \sum_{a \in A} \mathrm{Var}[X_a] \leq \sum_{a \in A} p_a = \mu\end{aligned}$$

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Let's apply this lemma to the estimate $|A^\star| \approx \frac{m}{t}|S_{h,t}(A^\star)|$ from our coordinated sampling.

Let $X = |S_{h,t}(A^\star)|$ and for $a \in A^\star$ let $X_a = [h(a) < t]$. Then $X = \sum_{a \in A^\star} X_a$ and for any $a, b \in A^\star$, X_a and X_b are independent. Also, let $\mu = \mathbb{E}_h[X] = \frac{t}{m}|A^\star|$.

Then for any $q > 0$,

$$\begin{aligned}
&\Pr_h\left[\left|\frac{\frac{m}{t}|S_{h,t}(A^\star)|}{|A^\star|} - 1\right| \geq q \cdot \frac{1}{\sqrt{\frac{t}{m}|A^\star|}}\right] \\
&\qquad = \Pr_h\left[\left||S_{h,t}(A^\star)| - \tfrac{t}{m}|A^\star|\right| \geq q \cdot \sqrt{\tfrac{t}{m}|A^\star|}\right] \\
&\qquad = \Pr_h\left[|X - \mu| \geq q \cdot \sqrt{\mu}\right] \\
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We needed strong universality in two places for this to work. Where?

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&\quad \leq \tfrac{1}{q^2}
\end{aligned}$$

We needed strong universality in two places for this to work. Where?

Application: Coordinated sampling

Let's apply this lemma to the estimate $|A^\star| \approx \frac{m}{t}|S_{h,t}(A^\star)|$ from our coordinated sampling.

Let $X = |S_{h,t}(A^\star)|$ and for $a \in A^\star$ let $X_a = [h(a) < t]$. Then $X = \sum_{a \in A^\star} X_a$ and for any $a, b \in A^\star$, X_a and X_b are independent. Also, let $\mu = \mathbb{E}_h[X] = \frac{t}{m}|A^\star|$.

Then for any $q > 0$,

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We needed strong universality in two places for this to work. Where? h must be uniform to get unbiased estimate, and pairwise independent for the lemma.

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Todays topic was hashing, and we have covered

- What is a random hash function, and what properties do we want.
- Two applications of universal hashing — unordered sets and signatures.
- Some concrete universal or strongly universal hash functions.
- An application of strongly universal hashing — coordinated sampling.
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NP-Completeness, part I

Christian Wulff-Nilsen
Advanced Algorithms and Data Structures
DIKU

December 12, 2022

Overview for today

- Problems and decision problems

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- Optimization problems (like SHORTEST-PATH) can usually be turned into decision problems (like PATH).

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- What is missing in this definition? Which encoding of the input is assumed?

Which encoding to pick?

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- These two ways of encoding k correspond to two different problems.

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- Encodings are always binary strings in our setting.

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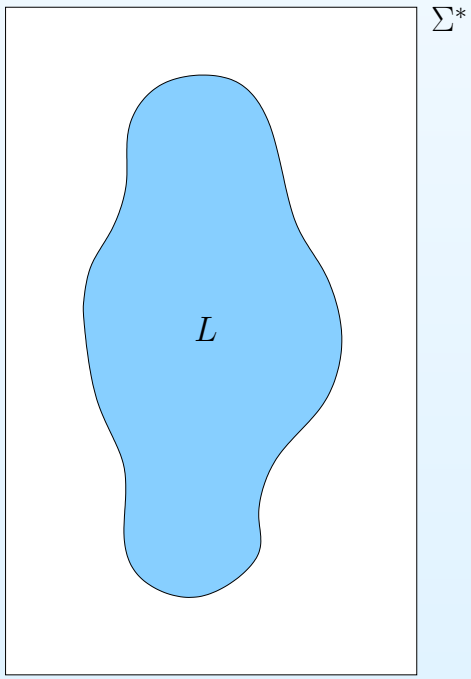
- Any language L over Σ is a subset of Σ^*.



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- For instance, `PATH` is the language of binary strings $\langle G, u, v, k \rangle$ where G is a graph, u and v are vertices of G, and there is a u-to-v path in G with at most k edges.

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- Deciding a language is stronger than accepting it.

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- We can now define the complexity class P:

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P in terms of acceptance

- Lemma:

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- $\subseteq$: straightforward.

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- Lemma:

$$P \stackrel{\text{def}}{=} \{L \subseteq \{0,1\}^* | \text{there exists an algorithm that decides } L \text{ in polynomial time}\}$$
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- $\subseteq$: straightforward.
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- Consider an algorithm A taking two parameters, $x, c \in \Sigma^*$.
- Instead of trying to find a solution to x (which may take long time), A instead *verifies* that c is a solution to x.

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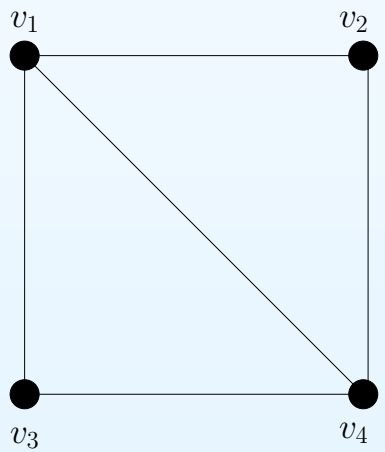

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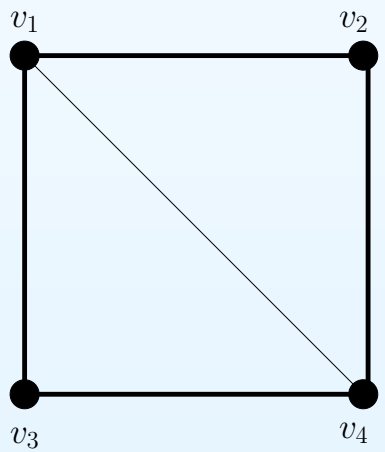

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- However, it is easy to show (next slide) that HAM-CYCLE can be verified in polynomial time.

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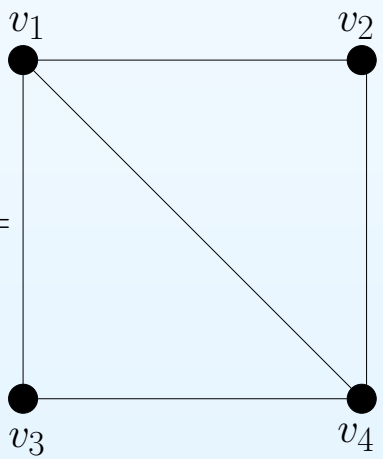
$G =$ 

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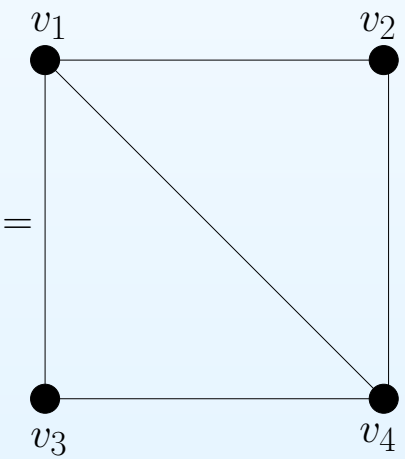
$G =$ 

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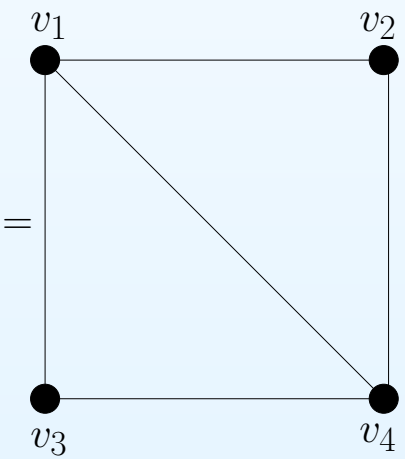
$G =$ 

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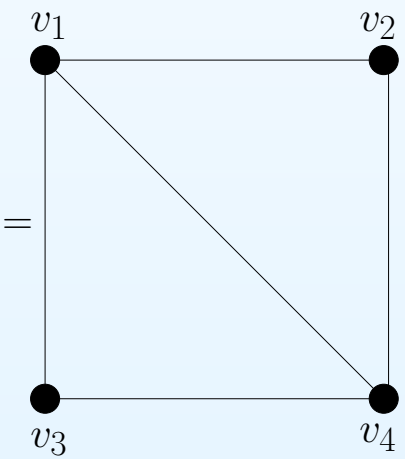
$G =$ 

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- Hence we can verify HAM-CYCLE in polynomial time.

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- Example:

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- What is known is that $P \subseteq$ NP $\cap$ co-NP.

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- To define NP-completeness, we need to first define polynomial-time reducibility.

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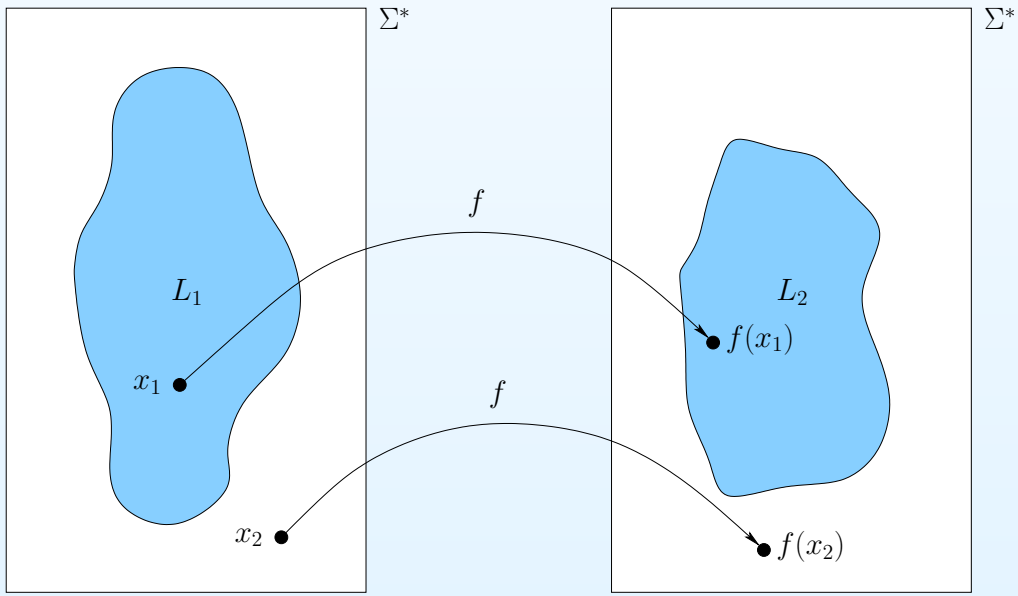

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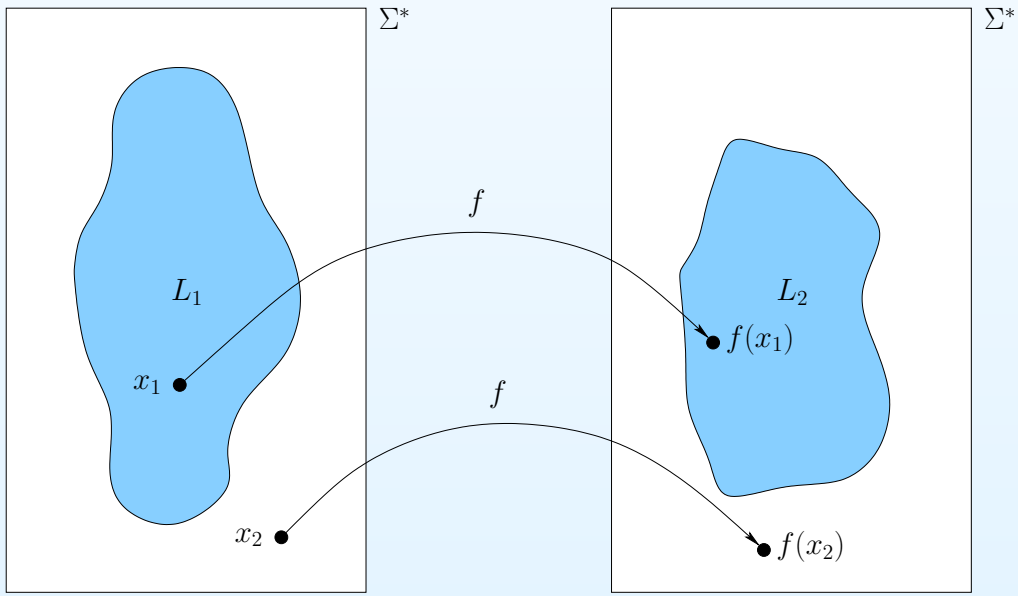

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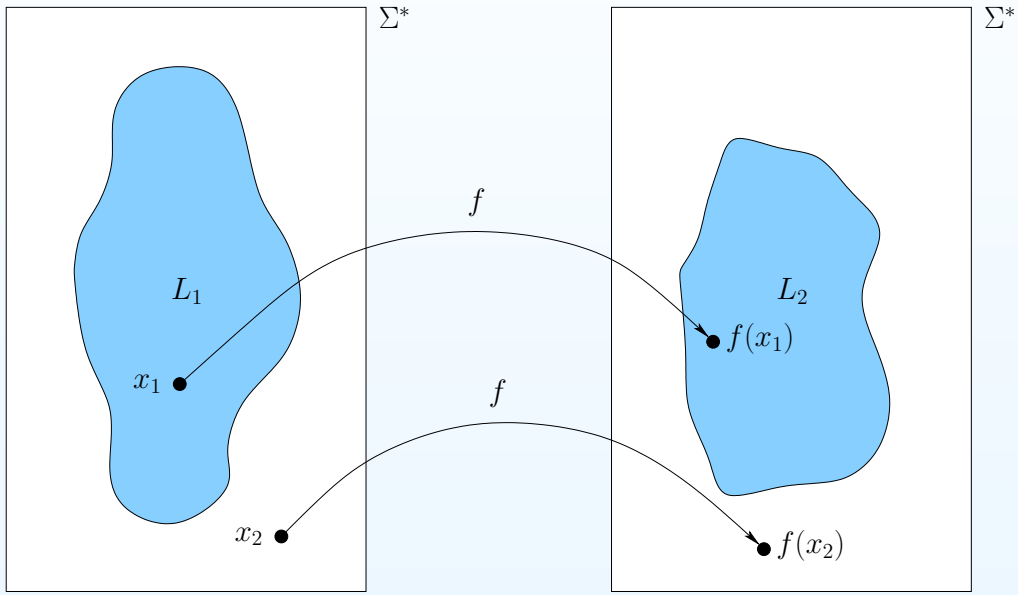

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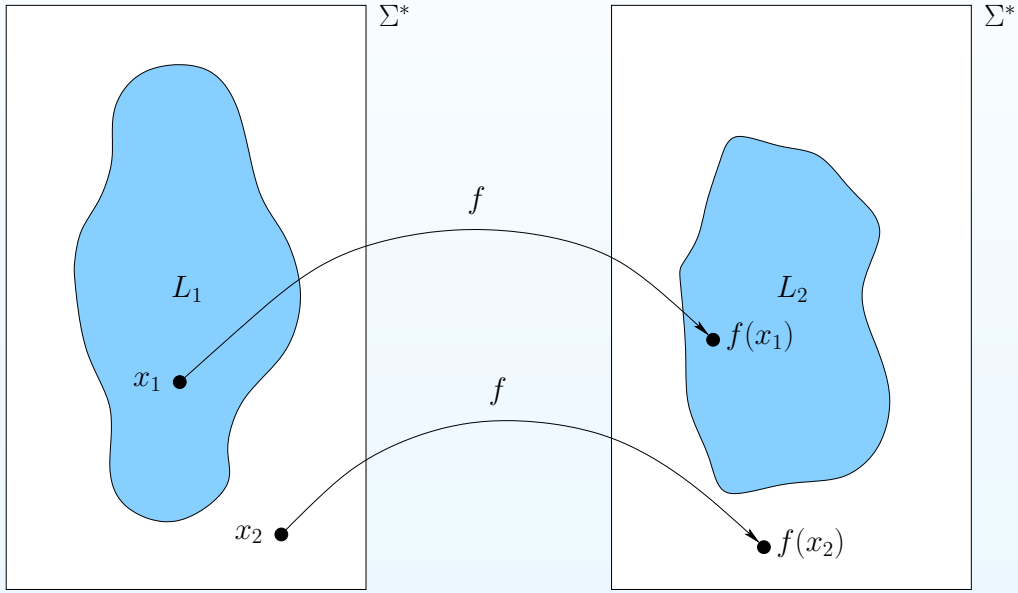

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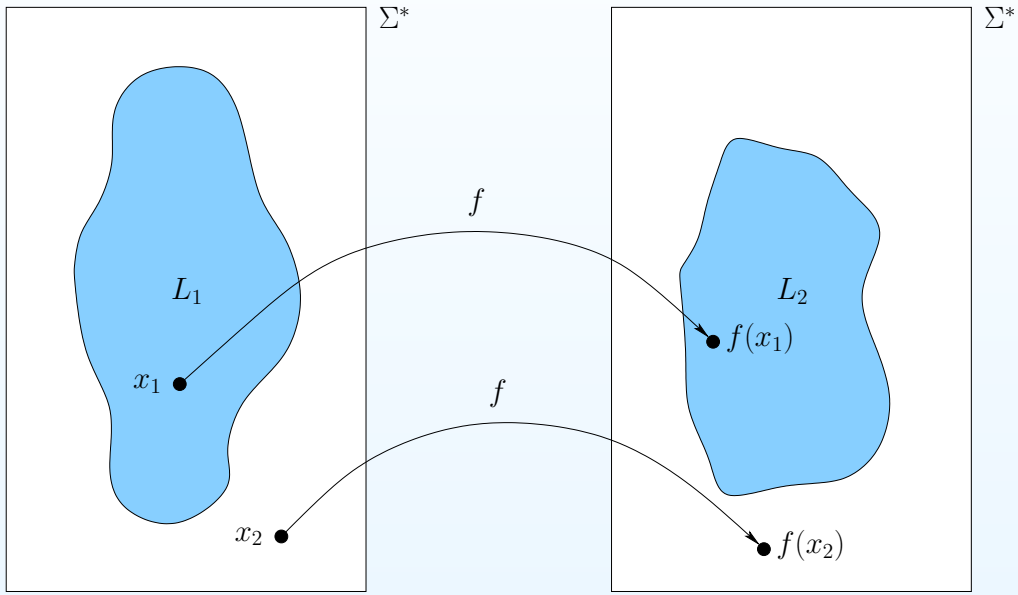

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- This follows since any instance x of L_1 can be solved by transforming it in polynomial time to an instance $y = f(x)$ of L_2 and then solving y with a polynomial-time algorithm for L_2.

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- We next show that the circuit satisfiability problem is NP-complete.

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- We can represent a circuit as an acyclic graph.

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- In other words, we need to find a polynomial-time algorithm A computing a function $f : \{0,1\}^* \to \{0,1\}^*$ such that

$$x \in L \Leftrightarrow f(x) \in \texttt{CIRCUIT-SAT}.$$

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- One bit of the last configuration $c_{T(n)}$ specifies the $0/1$-output of A.

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- The size of the circuit is still polynomial in n, however.

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- Thus, CIRCUIT-SAT is NP-hard.
- Since also CIRCUIT-SAT $\in$ NP, it follows that CIRCUIT-SAT is NP-complete.

Plan for next lecture

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- We also show that all these languages are in NP and hence they are NP-complete.

NP-Completeness, part II

Christian Wulff-Nilsen
Advanced Algorithms and Data Structures
DIKU

December 14, 2022

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- We can also view Q as a language L:

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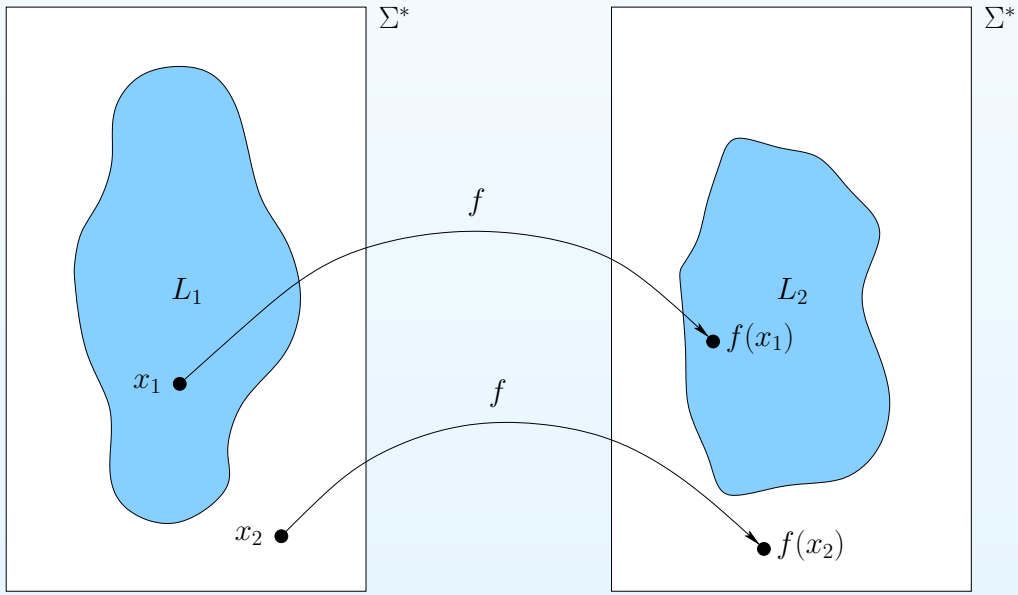

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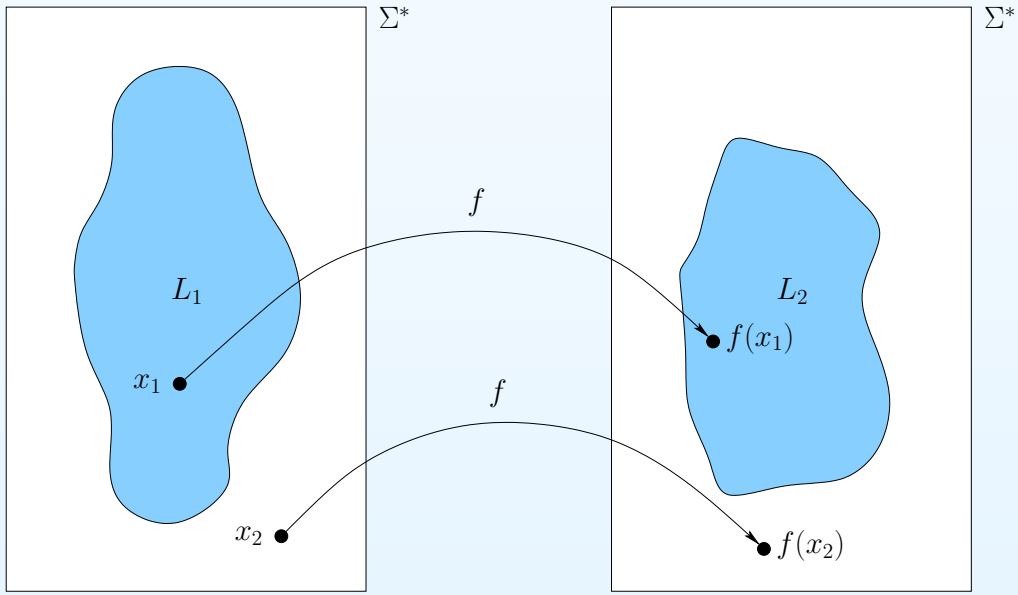

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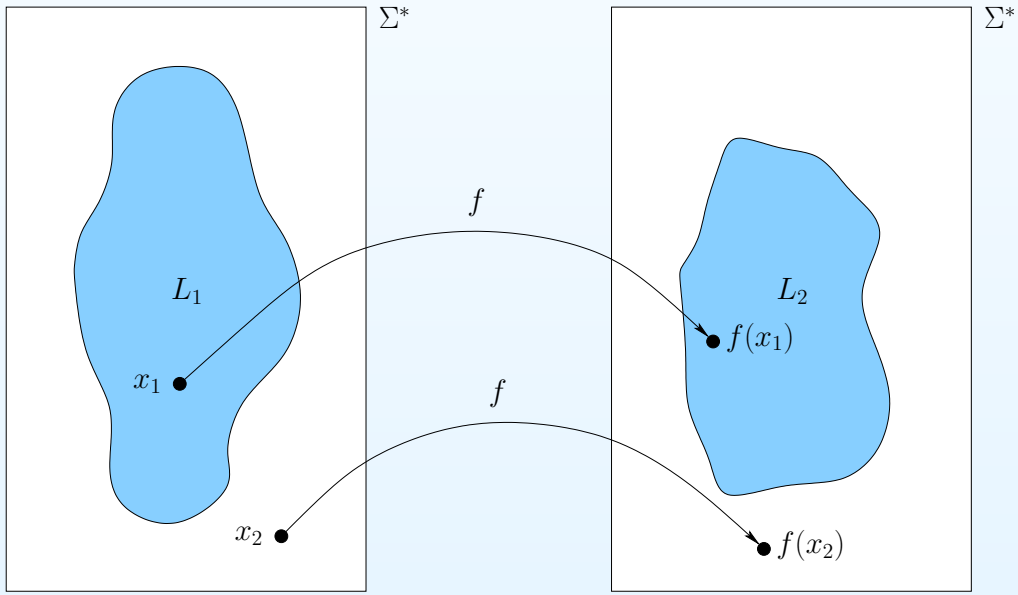

- We use the notation $L_1 \leq_P L_2$ for this.
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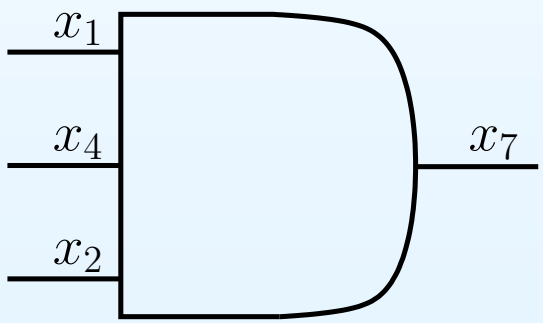

Sub-formula for gate: $x_7 \leftrightarrow (x_1 \wedge x_2 \wedge x_4)$

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- Example (see blackboard/Figure 34.10 in CLRS):

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- We have now shown that $\texttt{SAT}$ is NP-complete.

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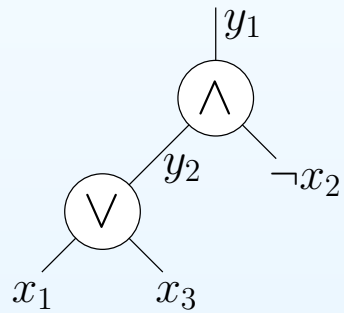

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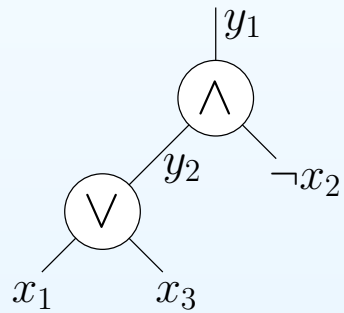

- We may assume that each node in the parse tree has at most two children. (why?)

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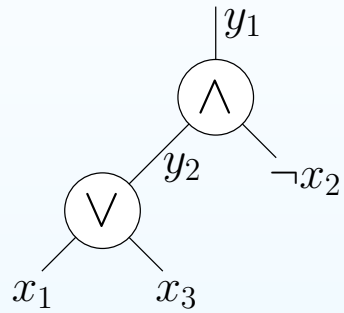

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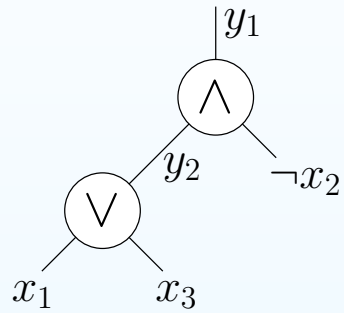

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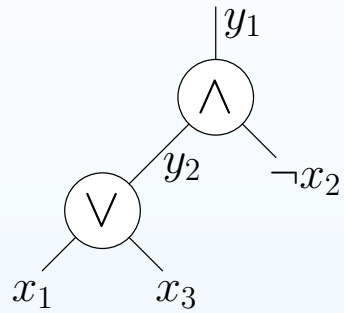

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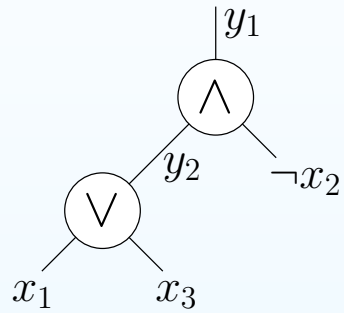

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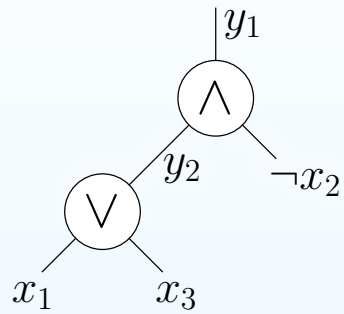

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- In words, $\neg\phi_i''$ is equivalent to the original ϕ_i'.

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- Thus, $\texttt{SAT} \leq_P \texttt{3-CNF-SAT}$ so $\texttt{3-CNF-SAT}$ is NP-complete.

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- To show NP-hardness, we reduce from $\texttt{3-CNF-SAT}$:

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Showing 3-CNF-SAT $\leq_P$ SUBSET-SUM

- Consider a 3-CNF-formula ϕ with n variables $x_1, \ldots, x_n$ and k clauses $C_1, \ldots, C_k$.

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- In other words, we want that ϕ is satisfiable if and only if S has a subset summing to t.

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- S consists of numbers $v_1, v'_1, v_2, v'_2, \ldots, v_n, v'_n$ and $s_1, s'_1, s_2, s'_2, \ldots, s_k, s'_k$.

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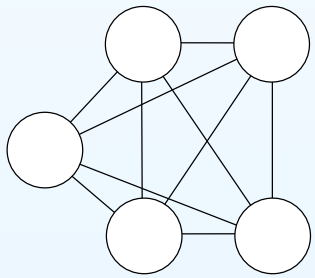

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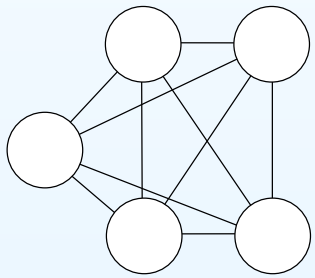

- The CLIQUE problem is the problem of determining if G contains a clique of a given size k:

$$\texttt{CLIQUE} = \{\langle G, k\rangle | G \text{ is a graph containing a clique of size } k\}.$$

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- To show CLIQUE $\in$ NP, consider an algorithm A taking two inputs, $\langle G, k \rangle$ and a certificate y.

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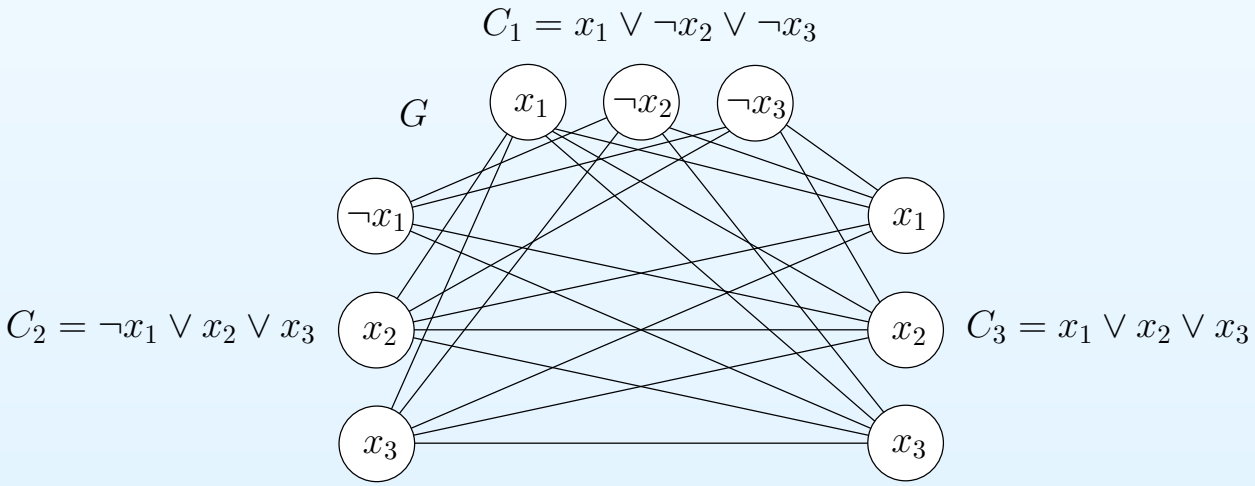

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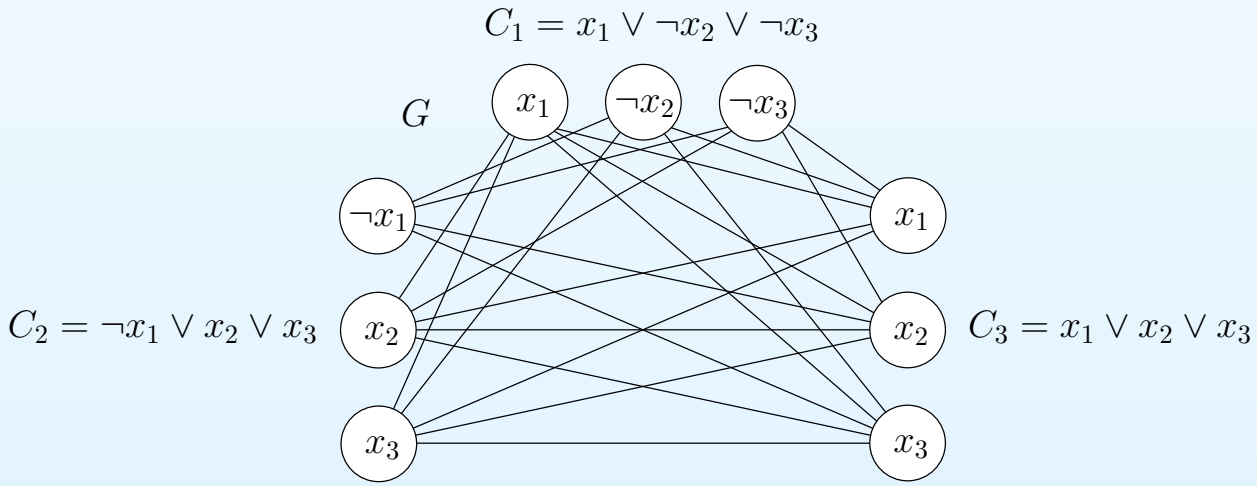

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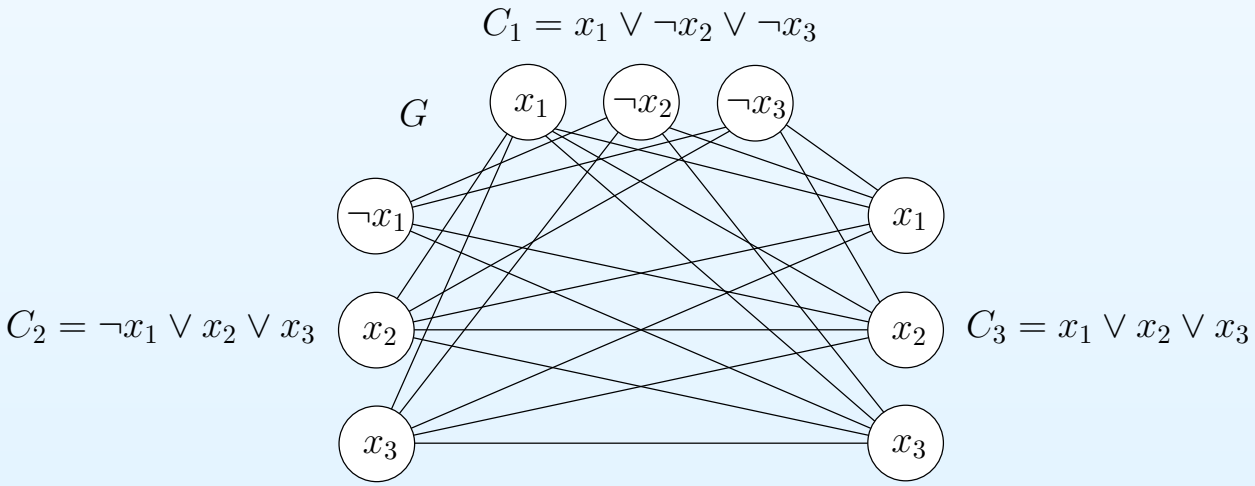

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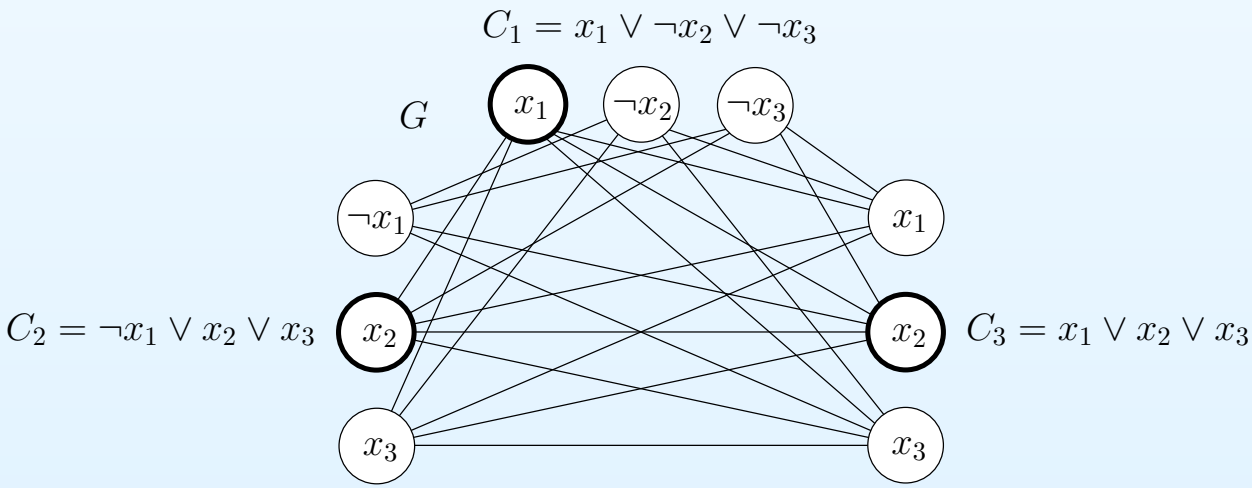
- Example with satisfying assignment $x_1 = 1, x_2 = 1, x_3 = 0$:



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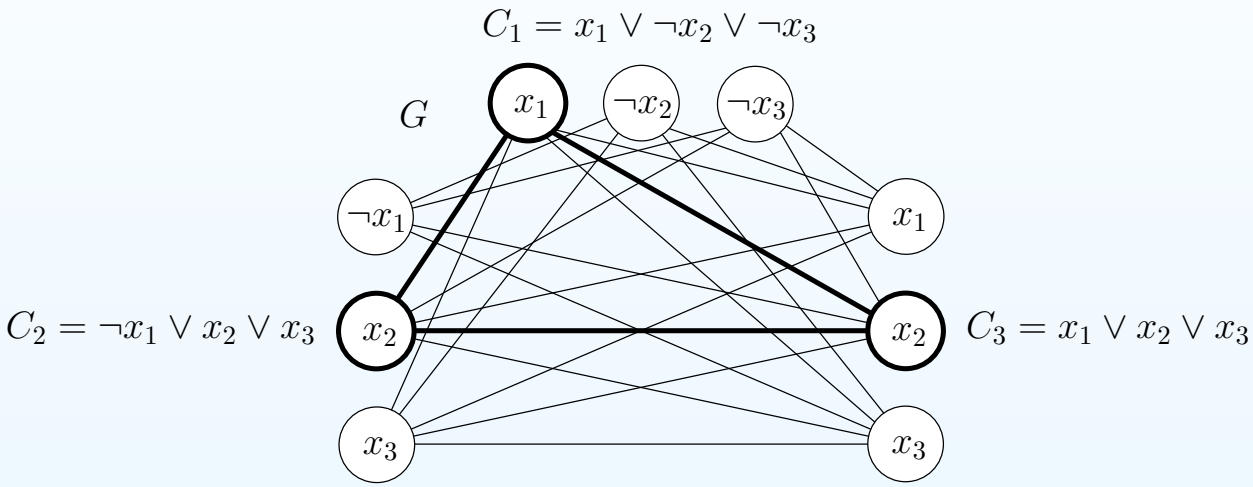

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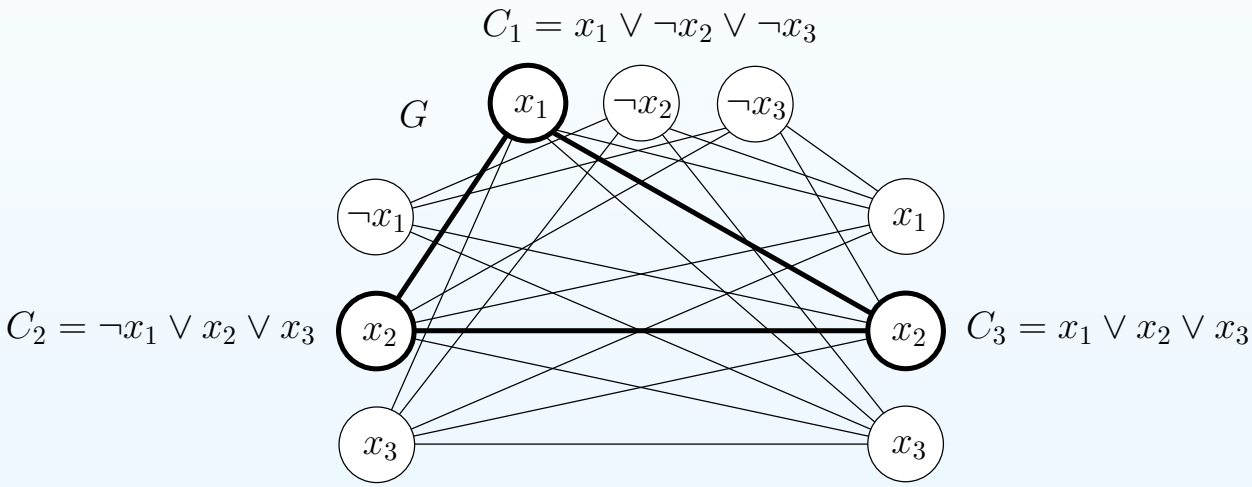

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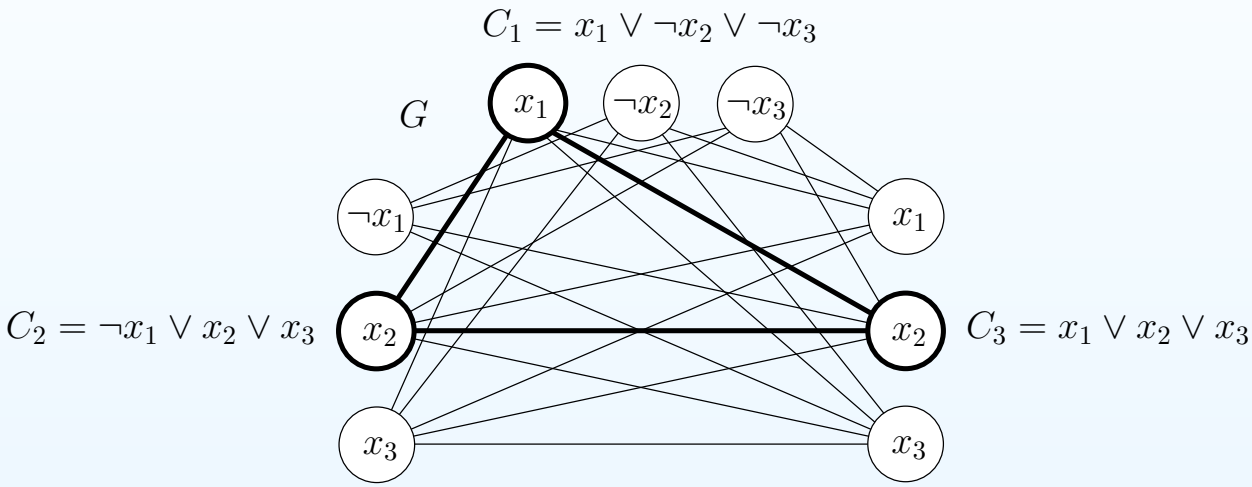

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- Hence, G has a clique of size k so $\langle G, k \rangle \in \text{CLIQUE}$.

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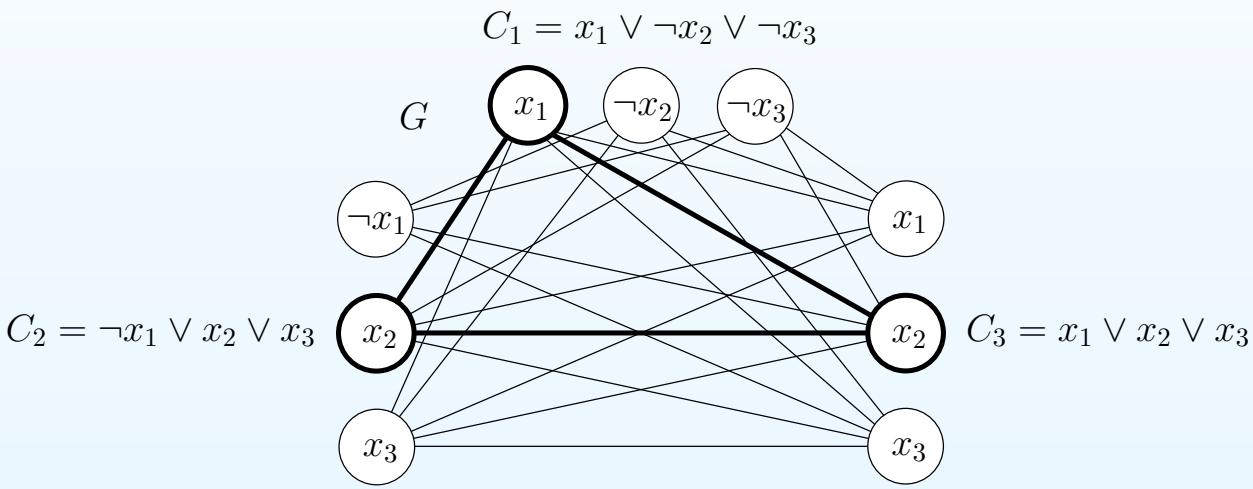

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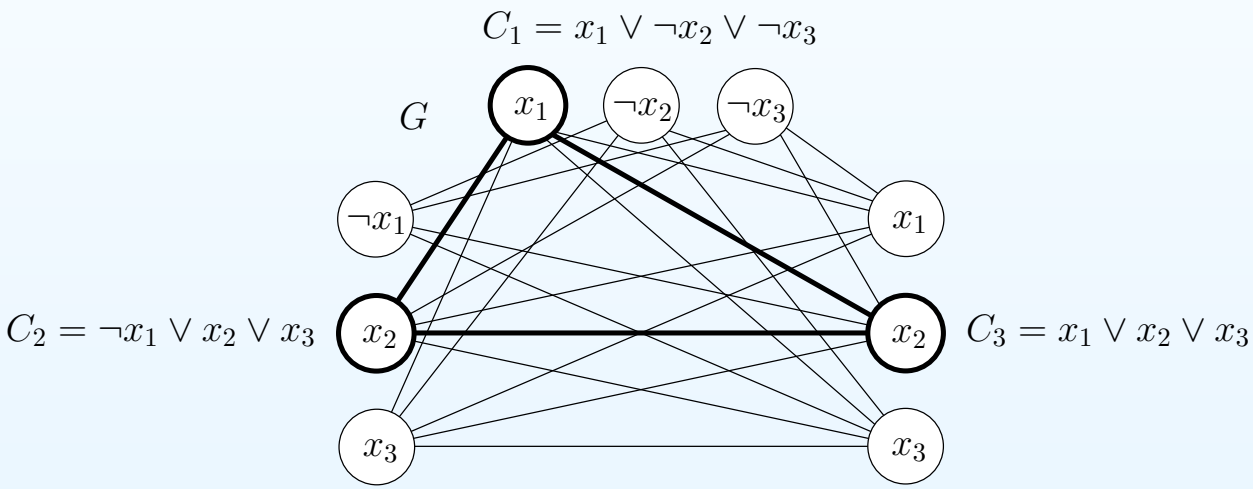

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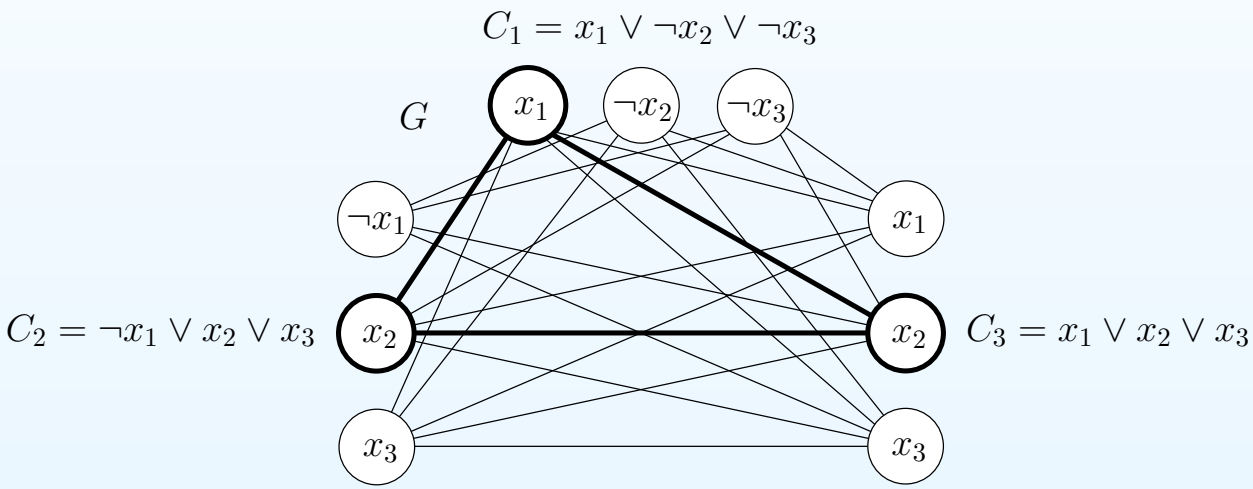

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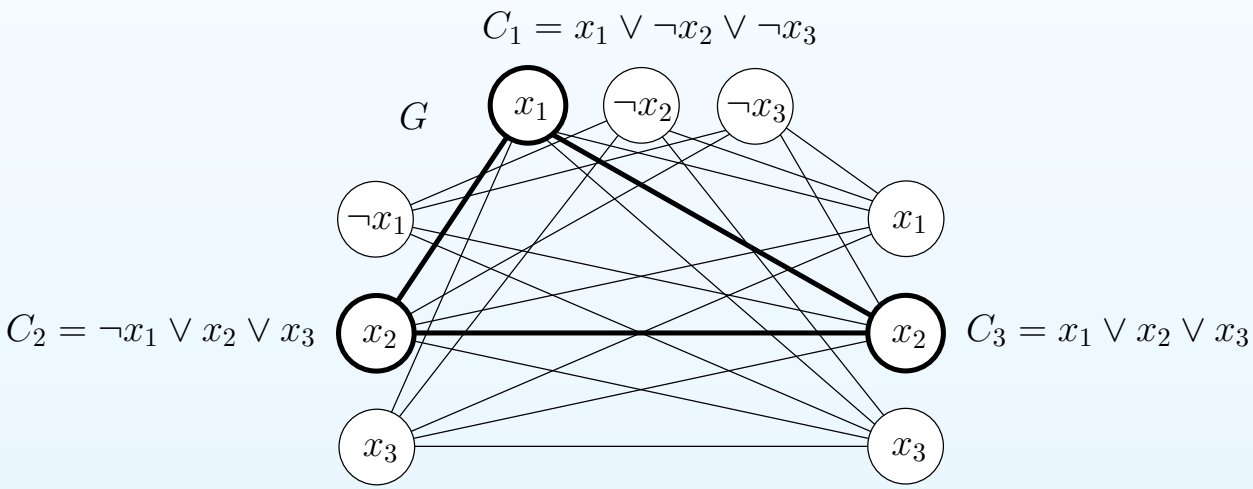

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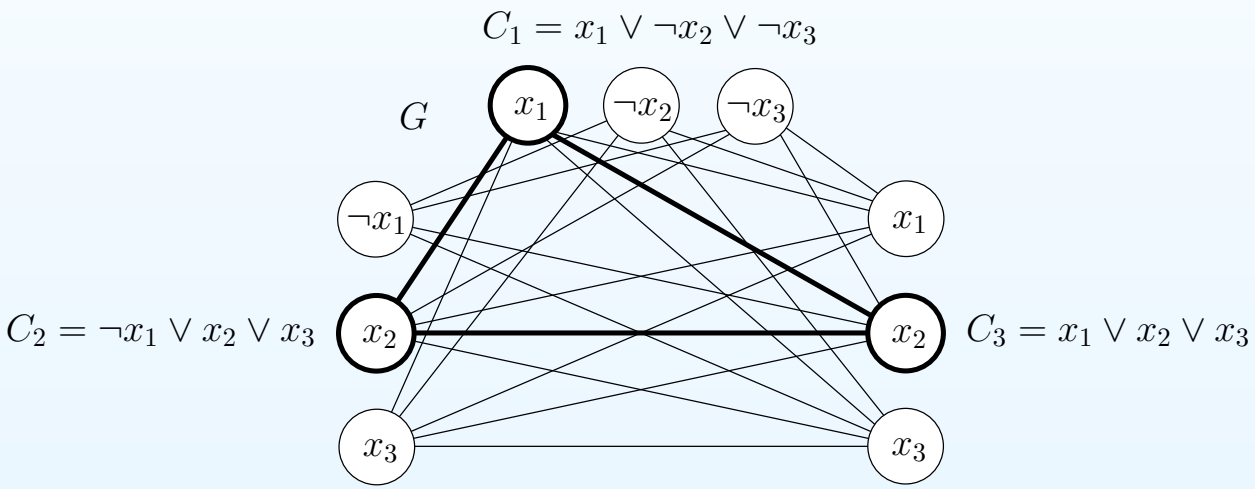

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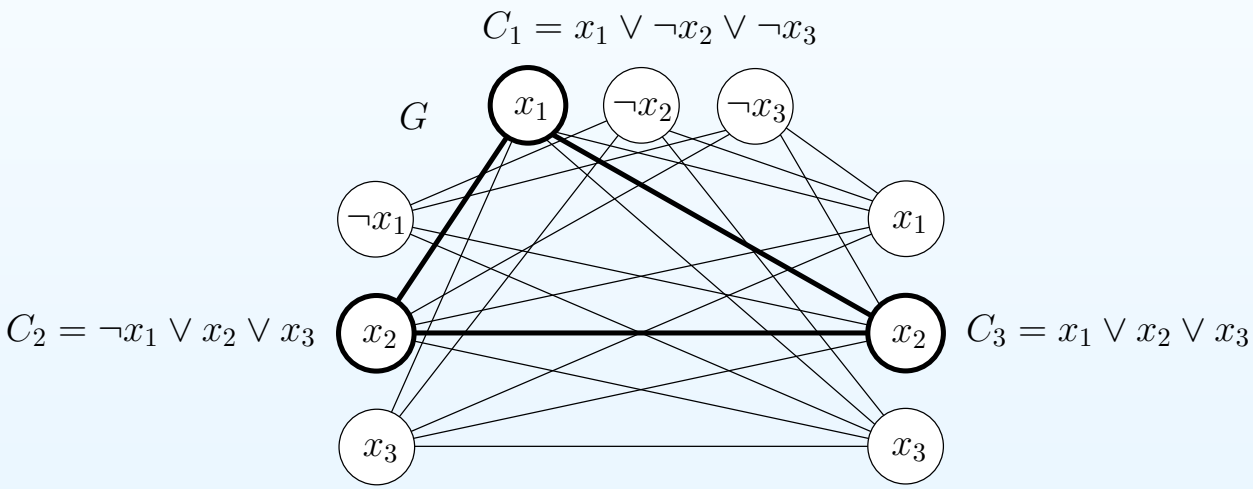

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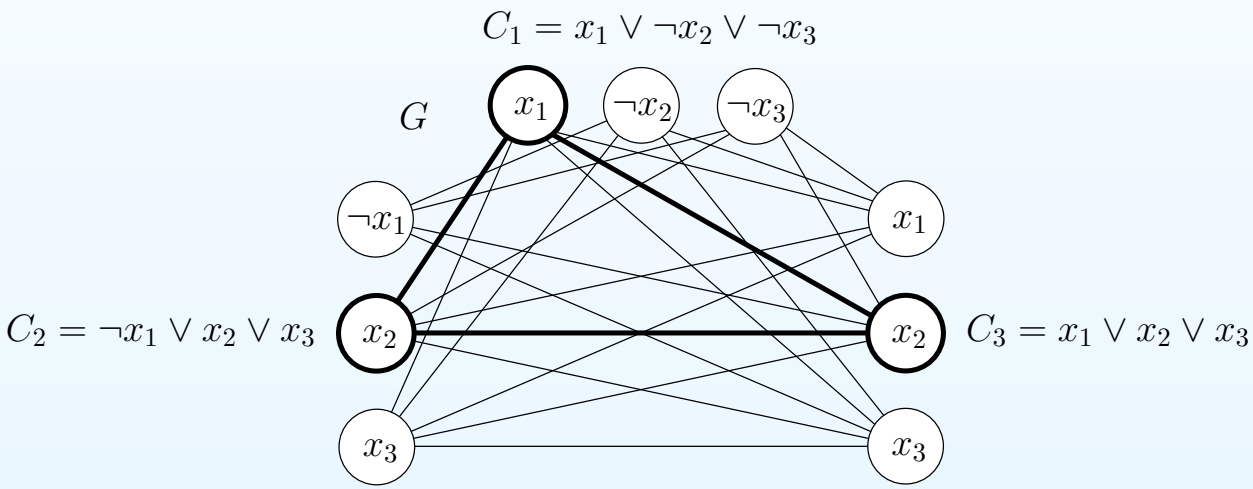

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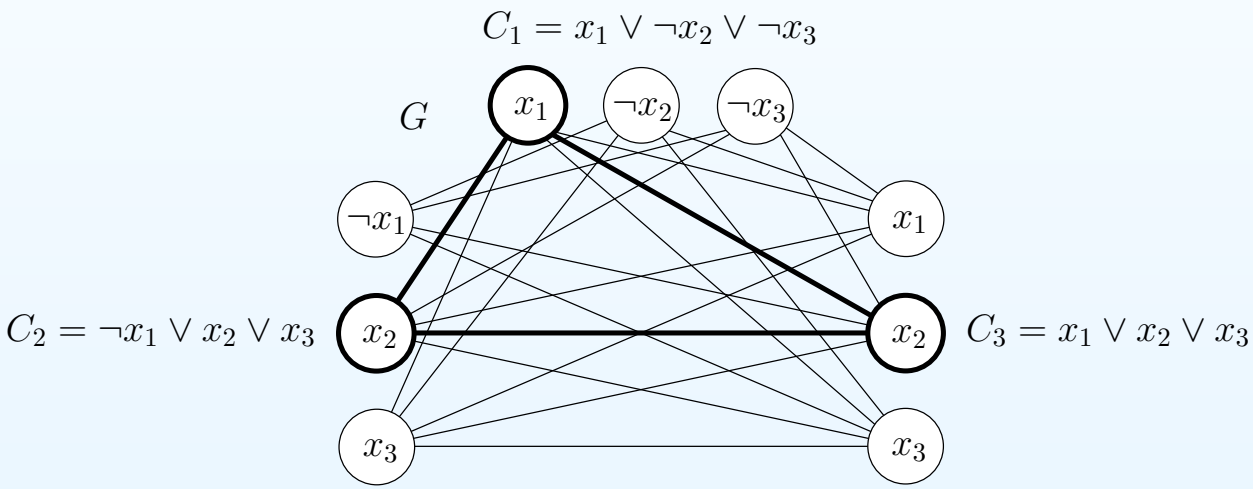

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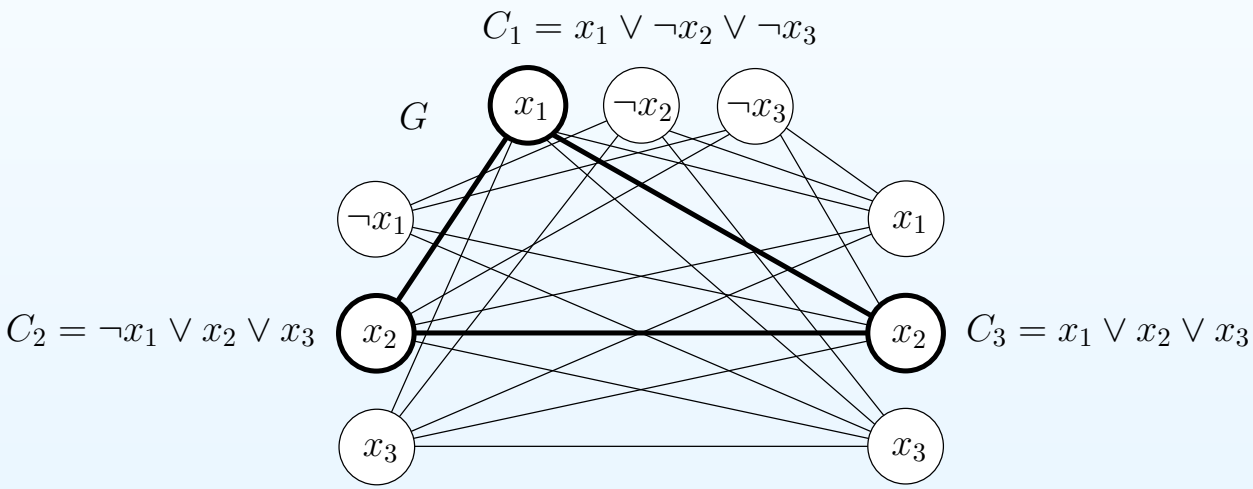

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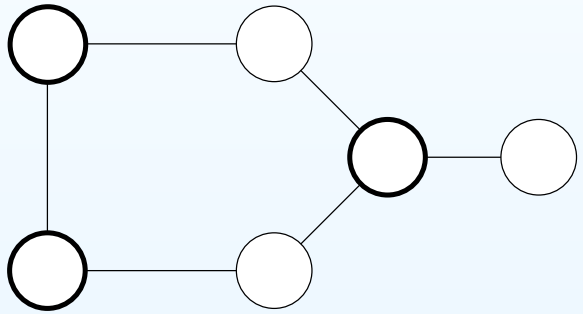

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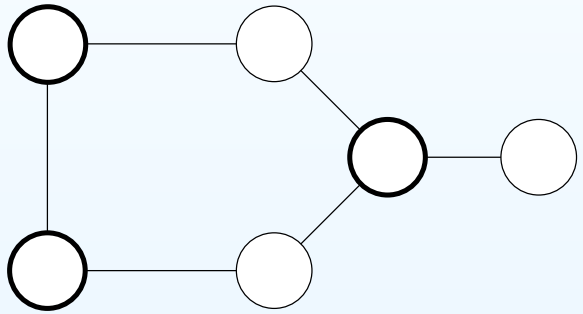

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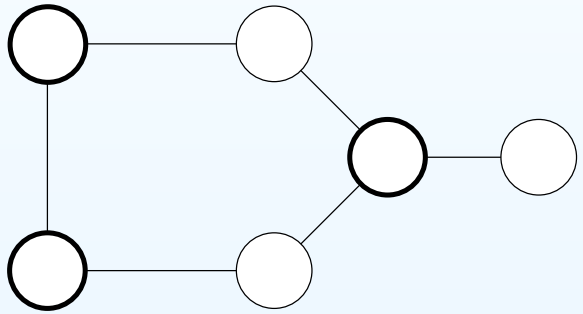

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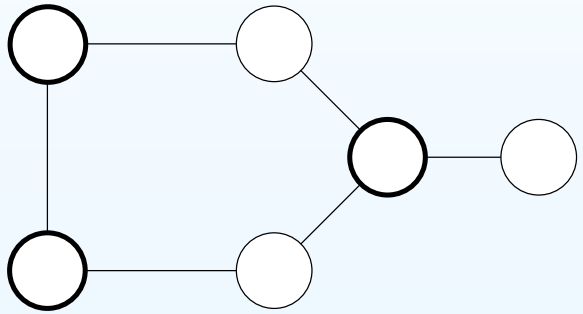

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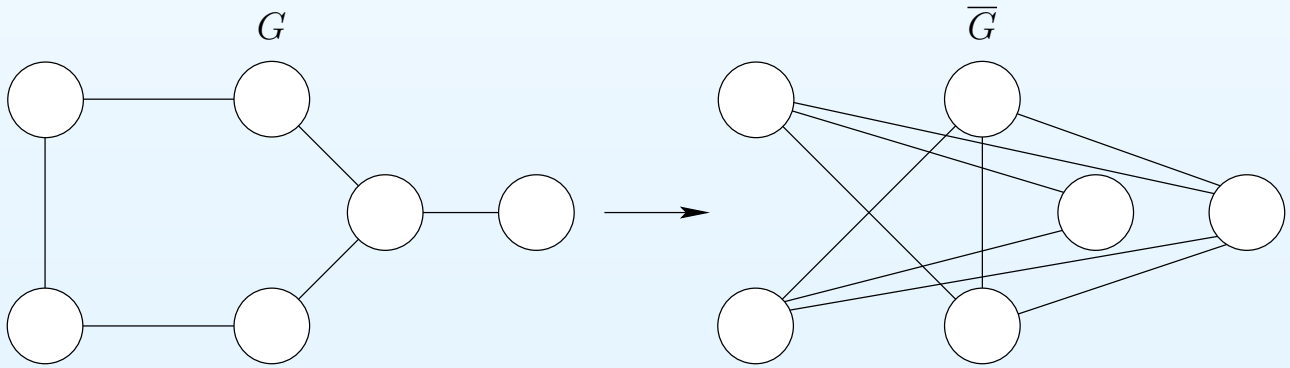

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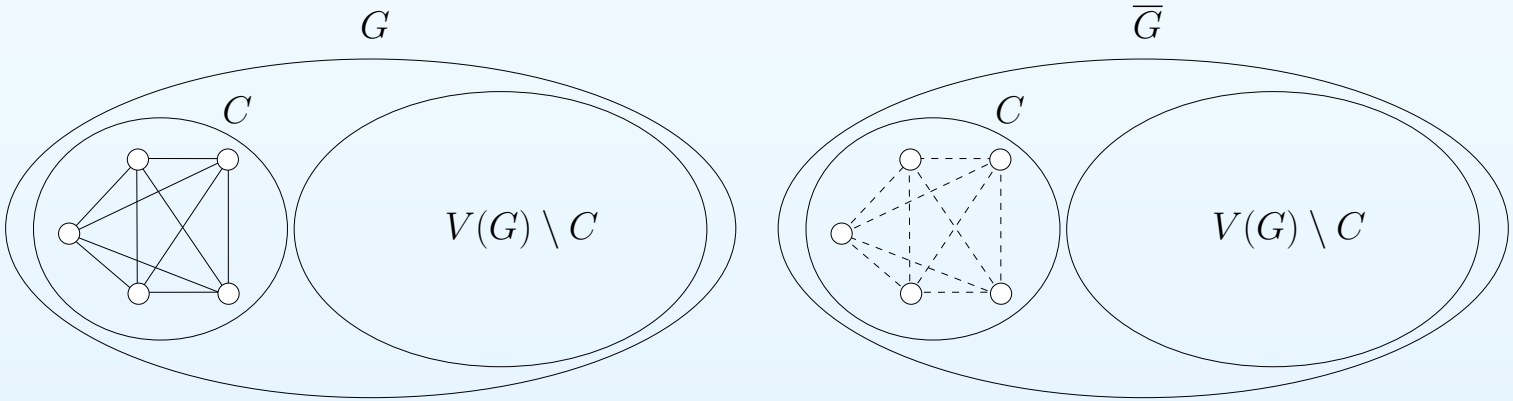

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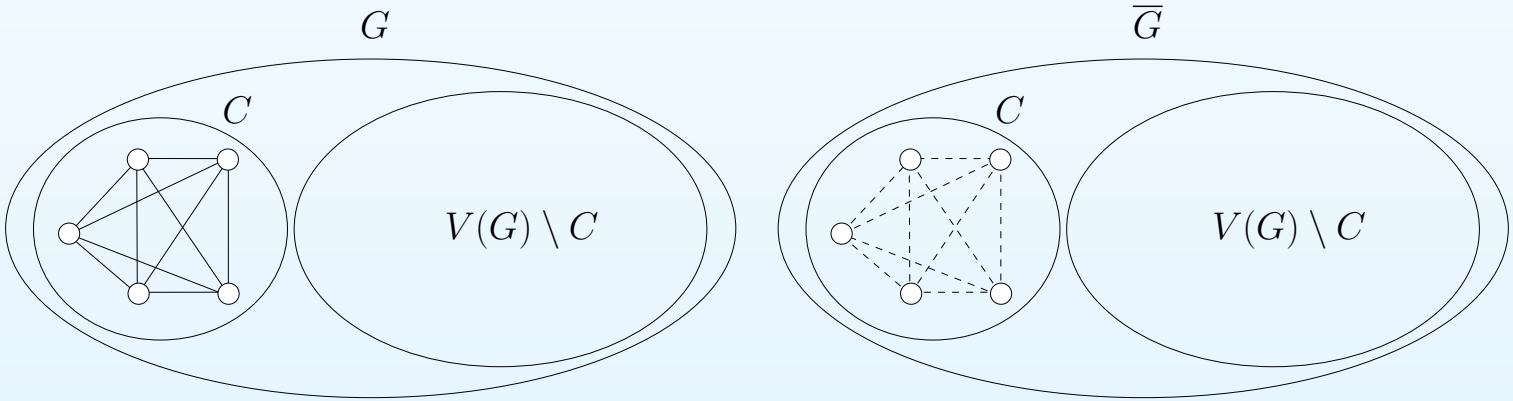

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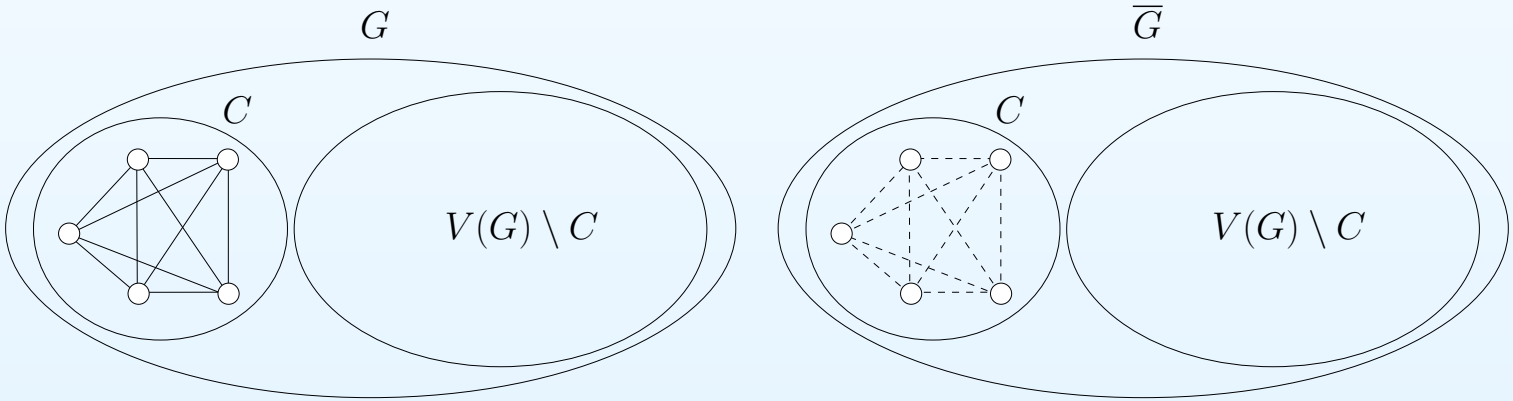

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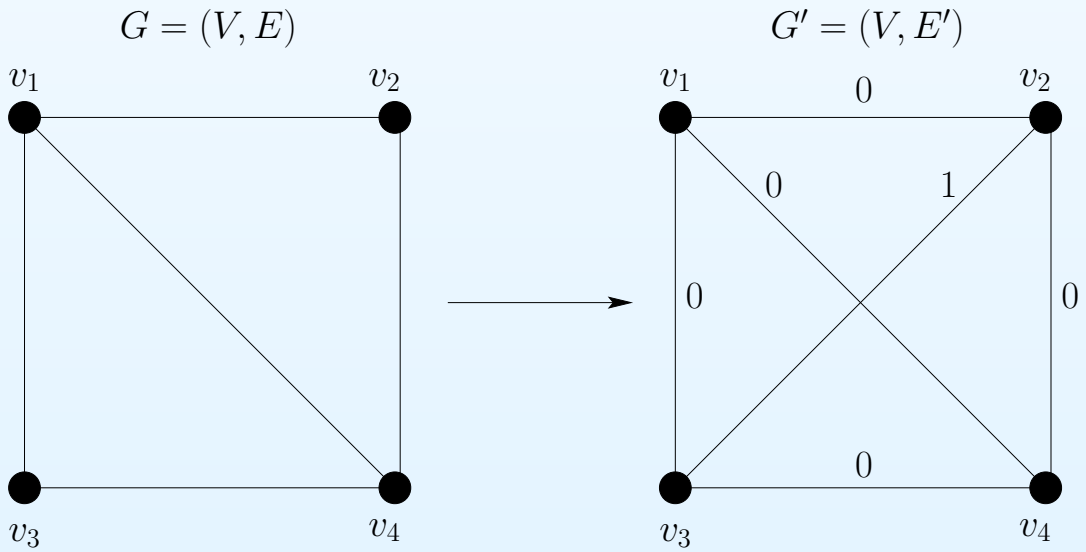

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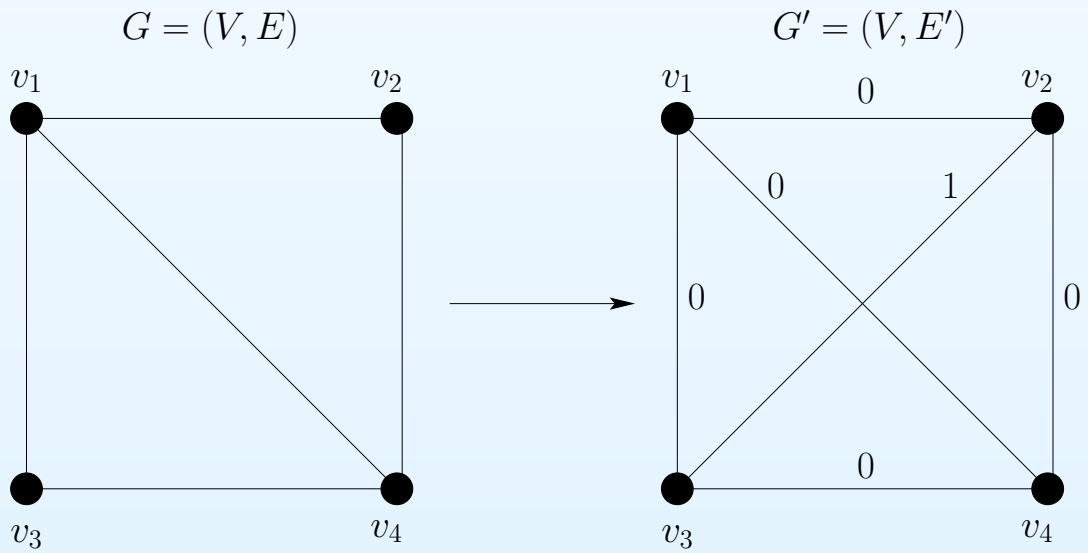

- Show that: $\langle G \rangle \in$ HAM-CYCLE $\Leftrightarrow \langle G', c, 0 \rangle \in$ TSP.

Advanced algorithms and data structures

Exact exponential algorithms and parameterized complexity

Jacob Holm (jaho@di.ku.dk)

December 16th 2024

Today's Lecture

Exact exponential algorithms and parameterized complexity

- Introduction
- Exact exponential algorithms
  - Exact TSP via Dynamic Programming
  - Dynamic Programming in general
  - Exact MIS via Branching
- Parameterized problems
  - "Bar fight prevention" aka k-Vertex Cover
  - Kernelization
  - Bounded search tree
- FPT vs XP
  - Example: Vertex k-Coloring
  - Example: k-Clique
  - Example: Clique parameterized by Δ
- Summary

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3) find an exact solution.

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: if we relax 1) to allow using exponential time.

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: if we relax 2) to only inputs with small fixed values of some parameter.

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: if we relax 3) to allow approximate solutions (final 2 lectures).

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AADS Lecture on EE/FPT, Part 2

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- A polynomial-time verifier $R(x, y)$; and
- a function $m(x) \in \mathcal{O}(\text{poly}|x|)$; such that
- for every problem instance x: x is a yes-instance if and only if there exists a certificate y of size $|y| \leq m(x)$ such that $R(x, y)$ is true.

Note: A certificate y is proof that x "has a solution", but y does not have to *be* a solution. However, a solution is often the most natural certificate.

Note: Every optimization problem has a decision version. What is it?

Every problem in NP has a simple brute-force algorithm of the following form: Given problem instance x, try all potential certificates y with $|y| \leq m(x)$ and check if $R(x, y)$ for any of them.

Since a potential certificate is just a bit string of length at most $m(x)$ there are at most $2^{m(x)}$ potential certificates to check, and each check takes $\mathcal{O}(\text{poly}|x|)$ time. Thus, if we assume $m(x)$ can be computed in $\mathcal{O}(\text{poly}|x|)$ time, the brute force running time is $\mathcal{O}(2^{m(x)} \cdot \text{poly}|x|)$.

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For any fixed $0 < a < b$, and $c, d \in \mathbb{R}$, we have $\mathcal{O}(a^n \cdot n^c) \subset \mathcal{O}(b^n \cdot n^d)$.

So when comparing exact exponential algorithms, the bases of the exponentials are much more important than the polynomial factors.

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$$f(n) \in \mathcal{O}^\star(g(n)) \iff \exists c \in \mathbb{R} : f(n) \in \mathcal{O}(n^c \cdot g(n))$$

In other words, $\mathcal{O}^\star(\cdot)$ is the same as $\mathcal{O}(\cdot)$ but ignores polynomial factors.

Notice that for all $0 < a < b$: $\mathcal{O}(a^n) \subset \mathcal{O}^\star(a^n) \subset \mathcal{O}(b^n)$.

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What do we mean by the "size" n of a problem? Typically:

n, or $m+n$	for graphs with n vertices and m edges.
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#variables	for SAT-type problems.

This measure of "size" is usually sufficient to describe the running time of the natural brute-force algorithm and to show improvements in better algorithms.

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SAT, MIS	$m(x) = n$	$T(n) \in \mathcal{O}^\star(2^n)$
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SAT, MIS	$m(x) = n$	$T(n) \in \mathcal{O}^\star(2^n)$	$\mathcal{O}^\star(1.45^n)$
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Problem: Given cities $c_1, \ldots, c_n$, and distances $d_{ij} = d(c_i, c_j)$, find tour of minimal length, visiting all cities exactly once. Equivalently, find permutation π minimizing $d(c_{\pi(n)}, c_{\pi(1)}) + \sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)})$.

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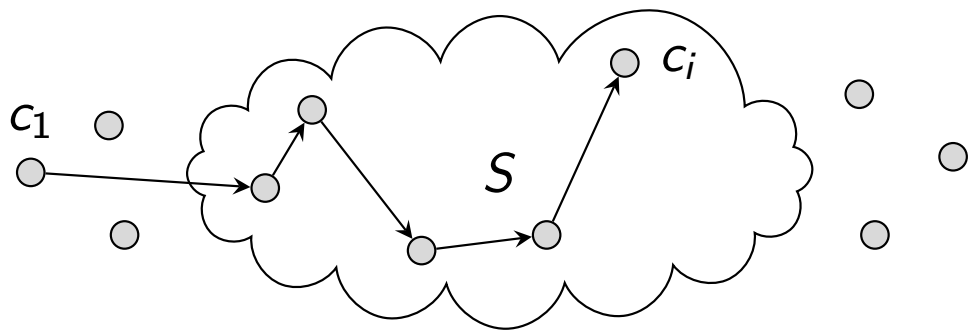

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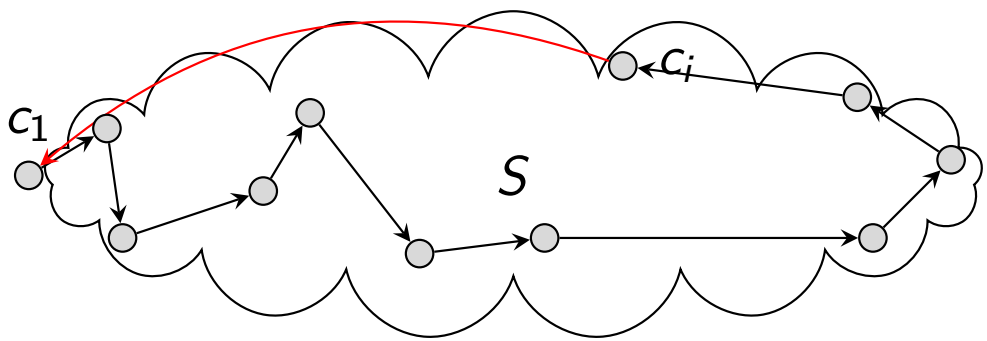

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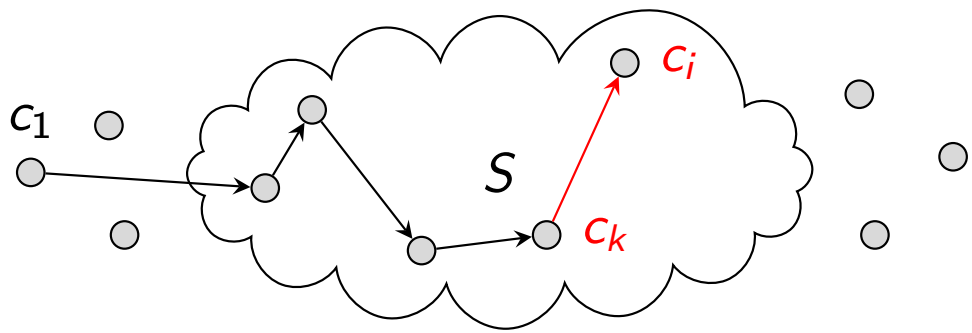

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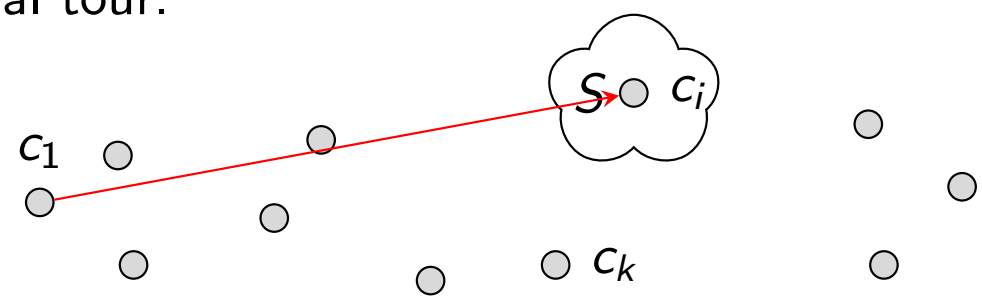

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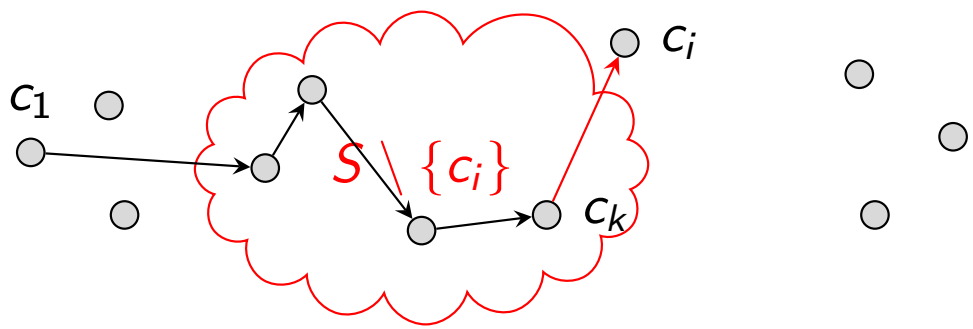

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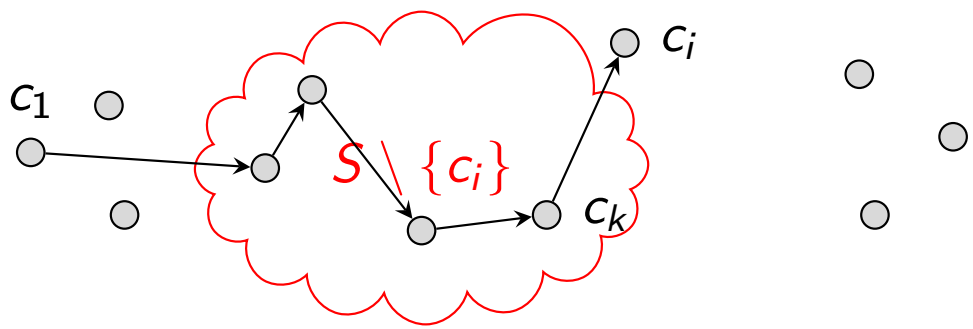

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And lines 3 and 8 compute only $n-1$ more path lengths each. □

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We can solve TSP by computing all $\mathrm{OPT}[S, c_i]$ values in order of increasing size of S.

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What is the running time of the algorithm? $\mathcal{O}^\star(2^n)$ if we assume additions take at most polynomial time in n. Much better than $\mathcal{O}^\star(n!)$.

Dynamic Programming in general

Similar to "Divide and Conquer" in that it requires "Optimal Substructure" but subproblems may be overlapping.

Instead of recursively solving smaller disjoint subproblems, "Dynamic Programming" solves all smaller subproblems in order of increasing size.

A hybrid idea called "Memoization" (not "Memorization"!) does the same by using recursion, but caching results so each subproblem is only solved once.

In our TSP example the original problem does not have the optimal substructure property (A piece of an optimal tour does not have to be an optimal tour of some subgraph). The trick is to notice that the problem of computing OPT$[\{c_2, \ldots, c_n\}, c_i]$ *does* have the property, and that TSP can be solved once we know that for all c_i.

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Exact MIS via Branching

Problem: Given undirected graph (V, E), find the maximum cardinality of $I \subseteq V$ so each edge has at most one endpoint in I.

Such a set I is called a *Maximum Independent Set (MIS)* for the graph.

Naive: Try all 2^n subsets (where $n = |V|$). This takes $\mathcal{O}^\star(2^n)$ time.

For $v \in V$ define $N[v] := \{v\} \cup \{w \in V \mid (v, w) \in E\}$. This is called the *closed neighborhood* of v.

Observation: $N[v] \cap I \neq \emptyset$ for all $v \in V$ and all MIS I.
Why?

```
1: function MISsize(G = (V, E))
2:     if V = ∅ then return 0
3:     v ← vertex in V of minimum degree.
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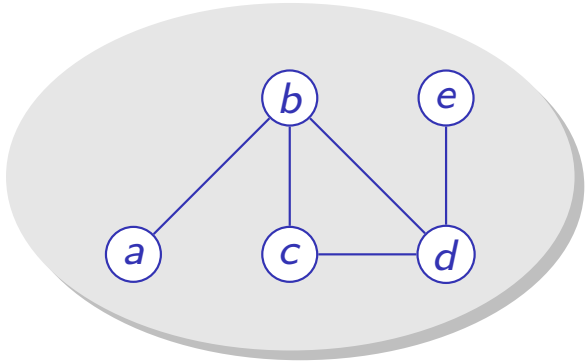

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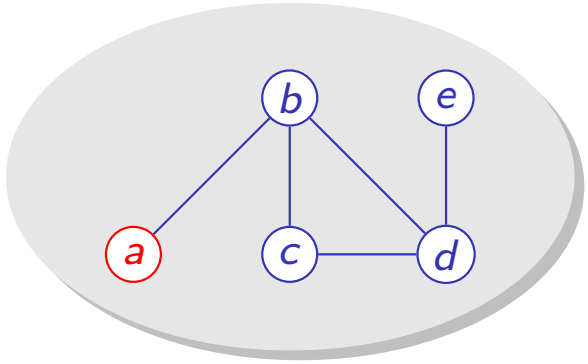
$v = a$



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$v = a$

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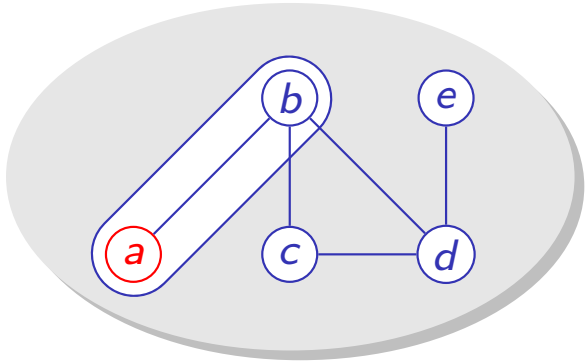

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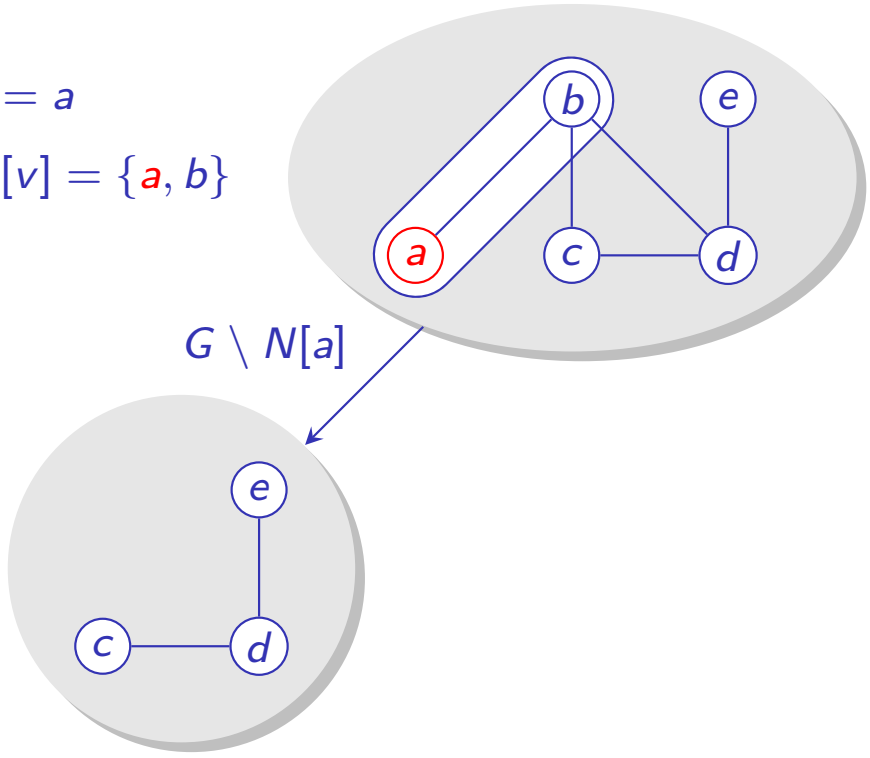

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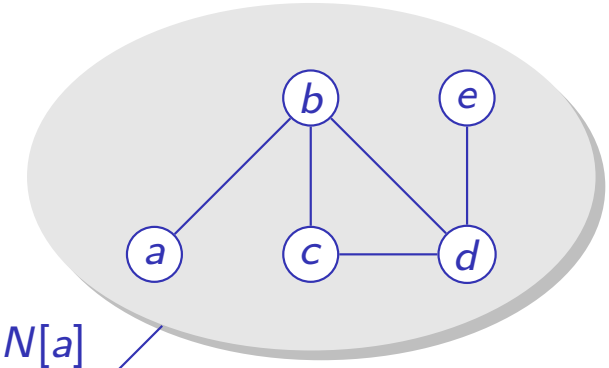

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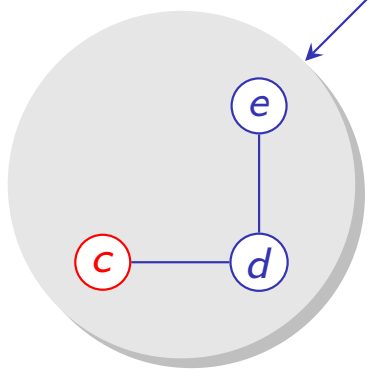

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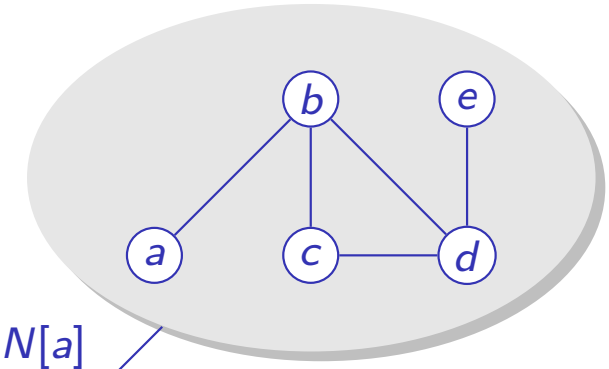

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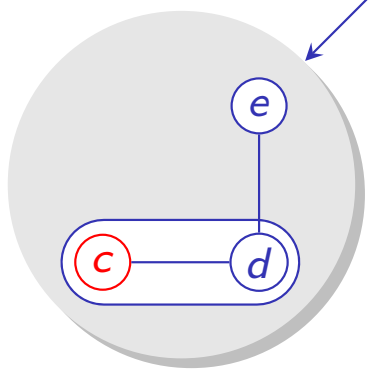

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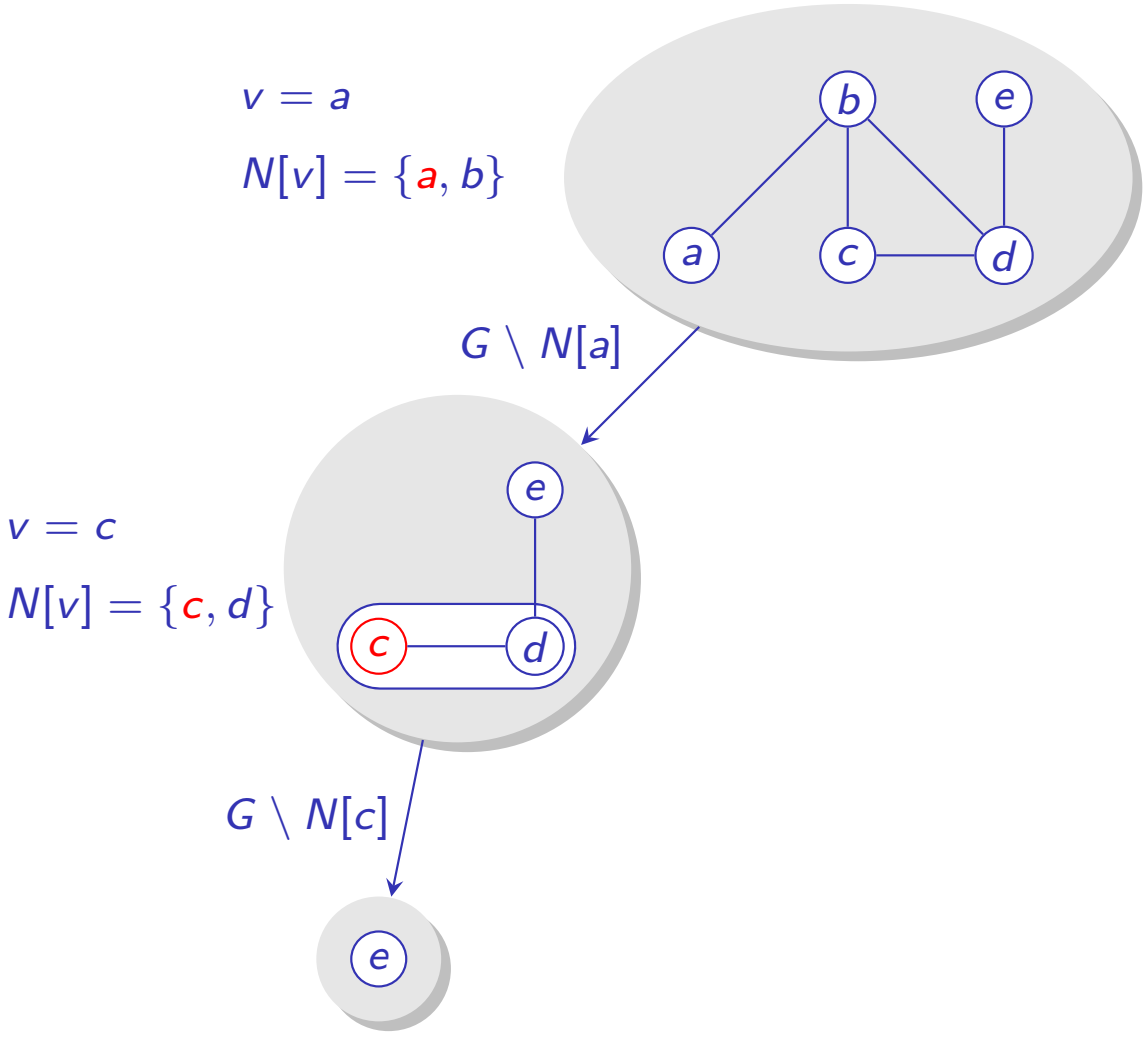

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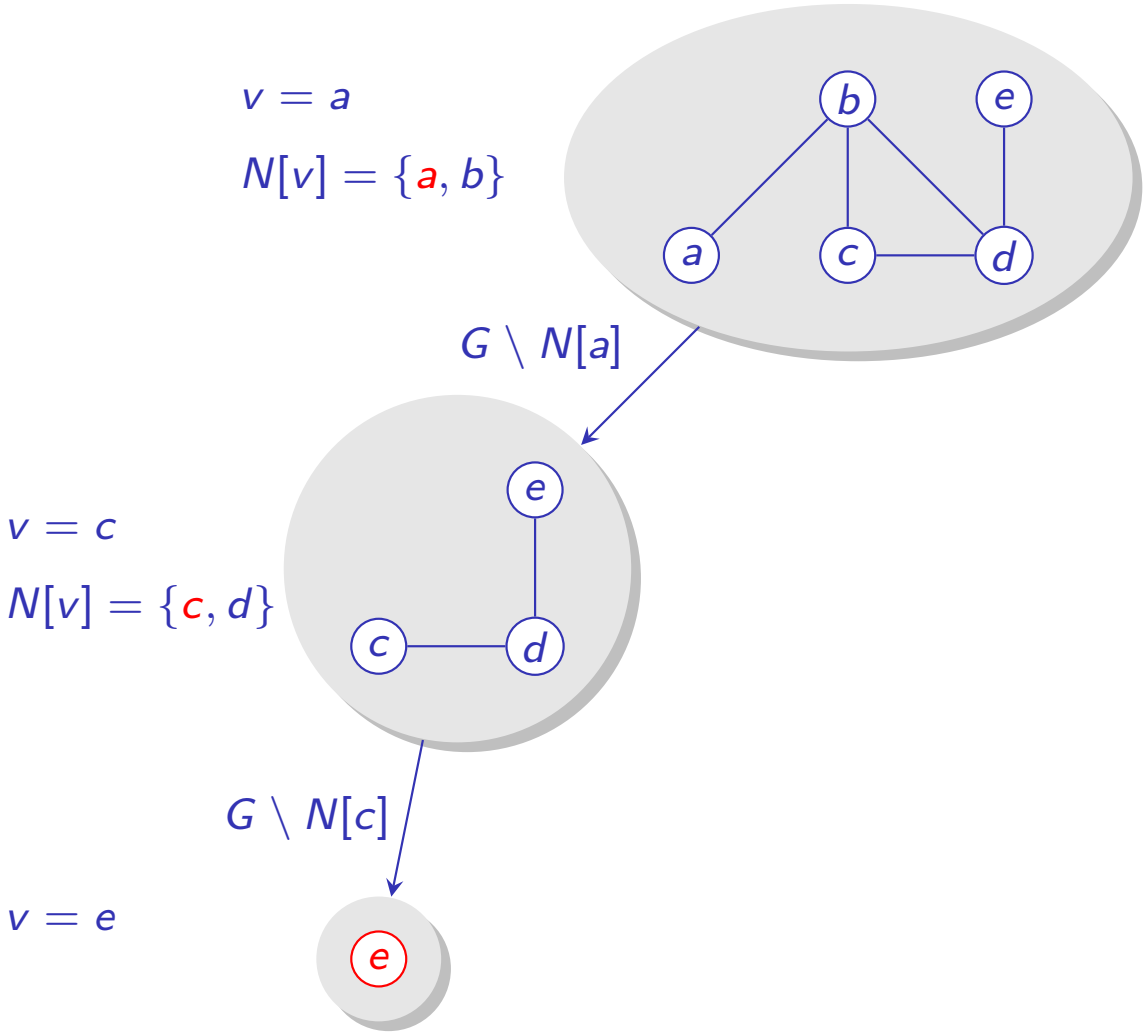

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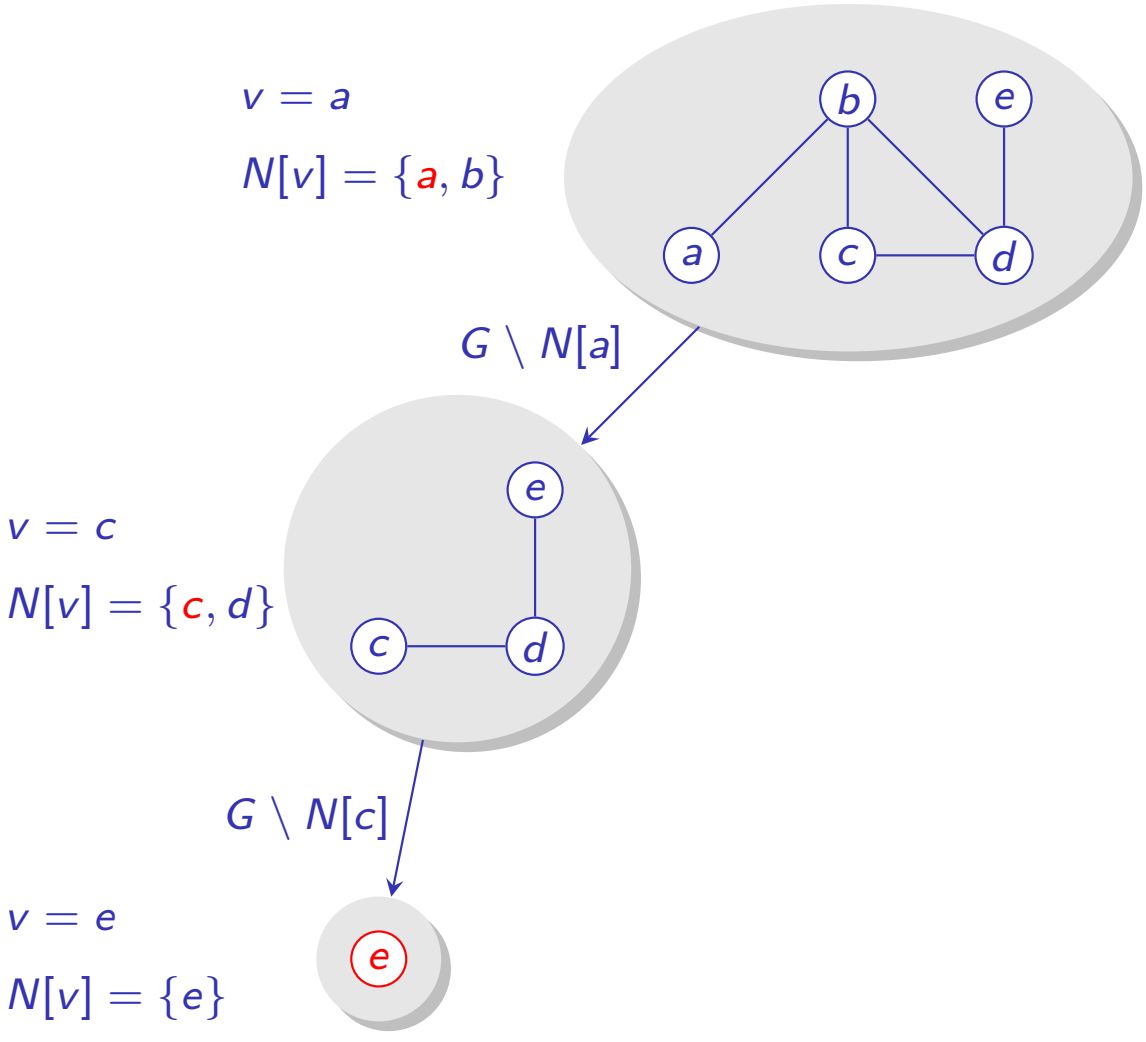

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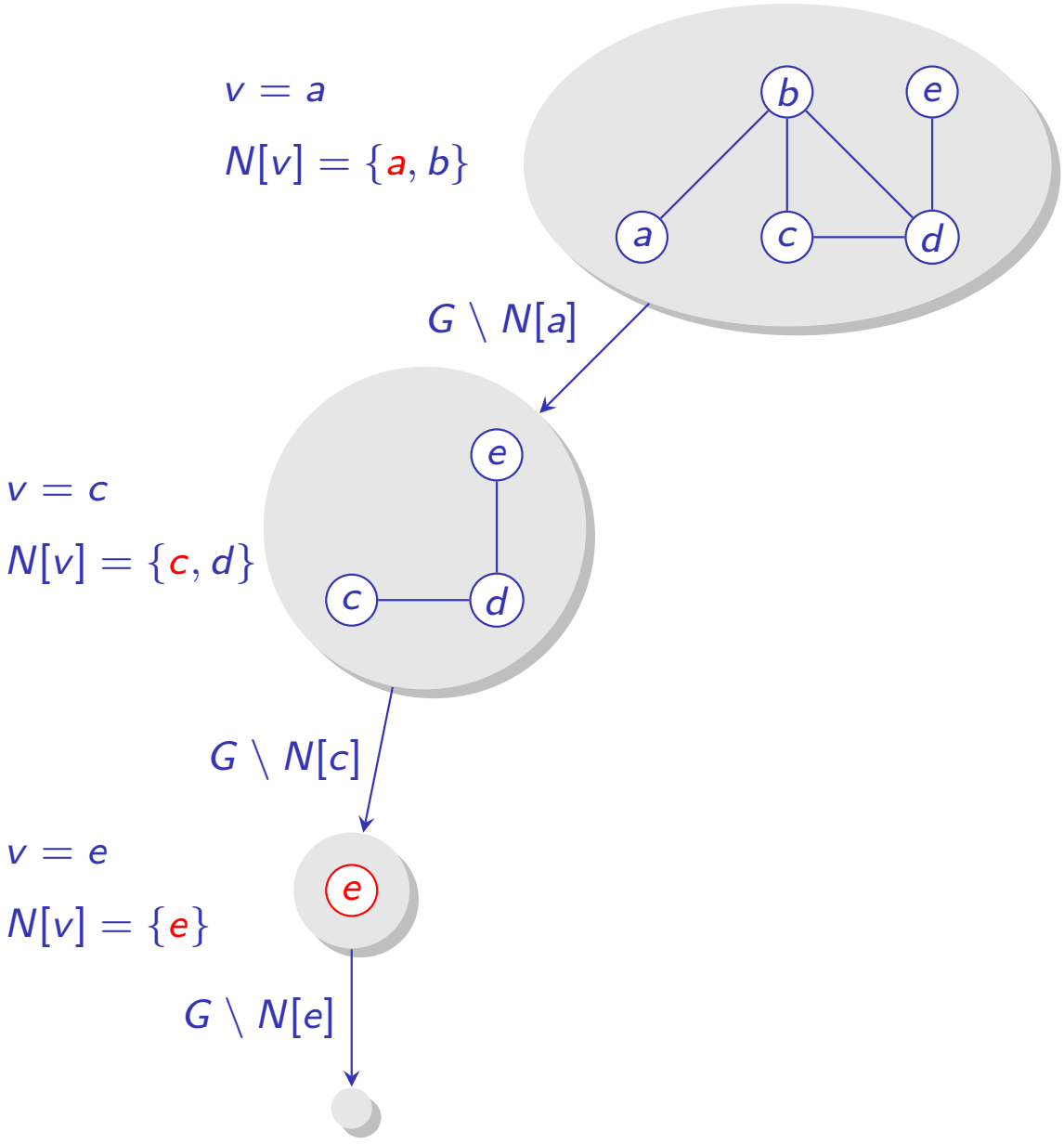

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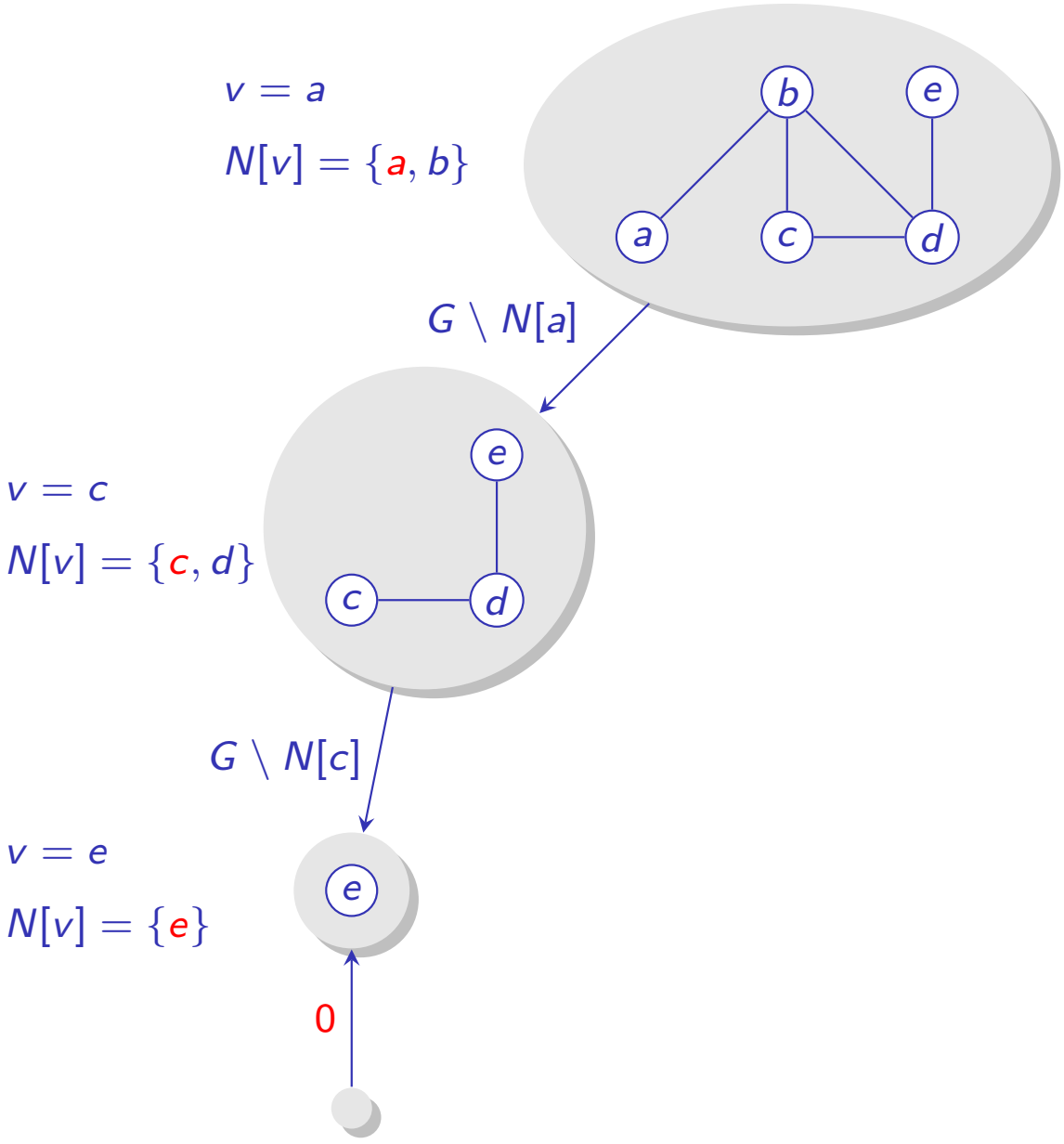

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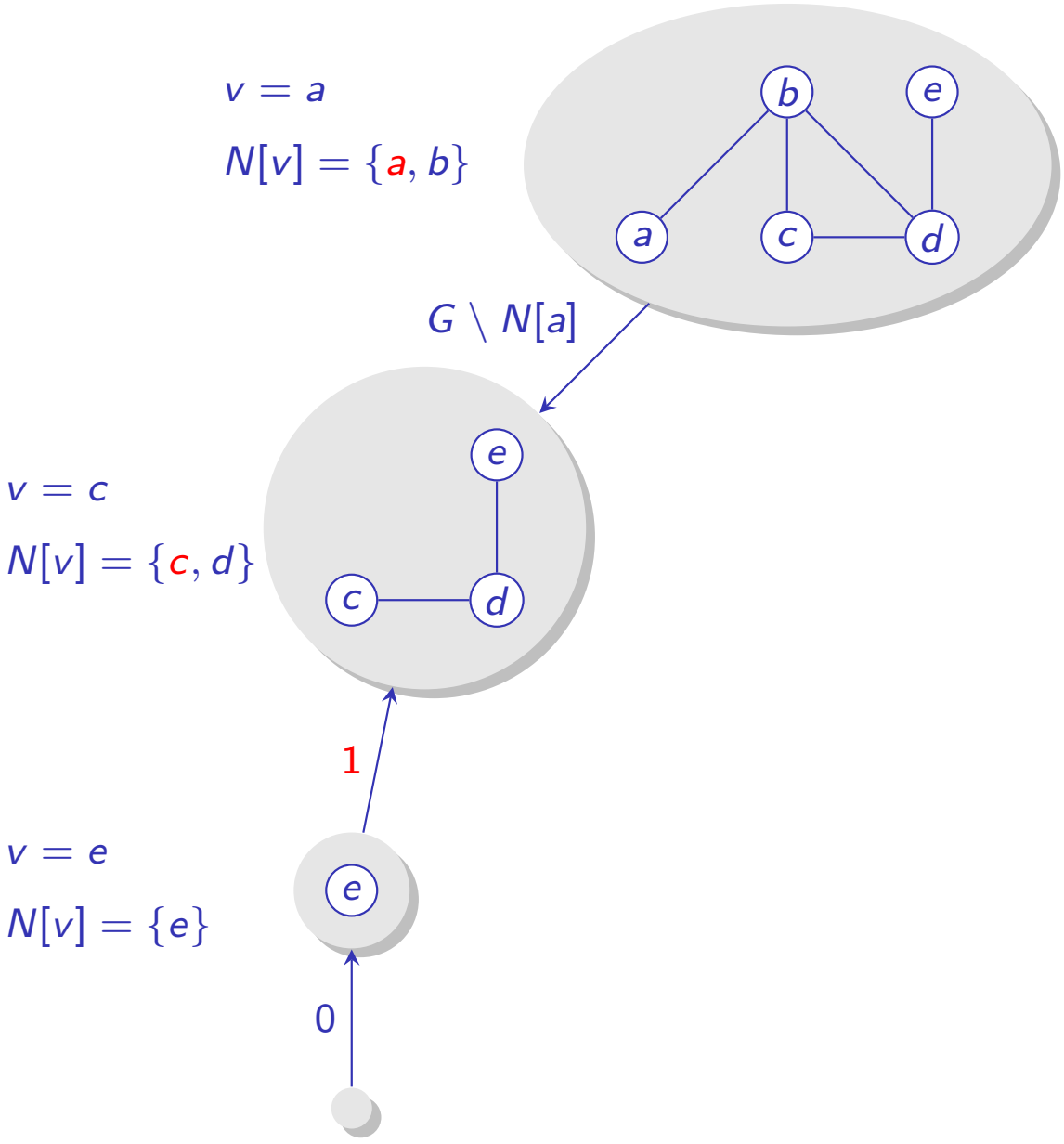

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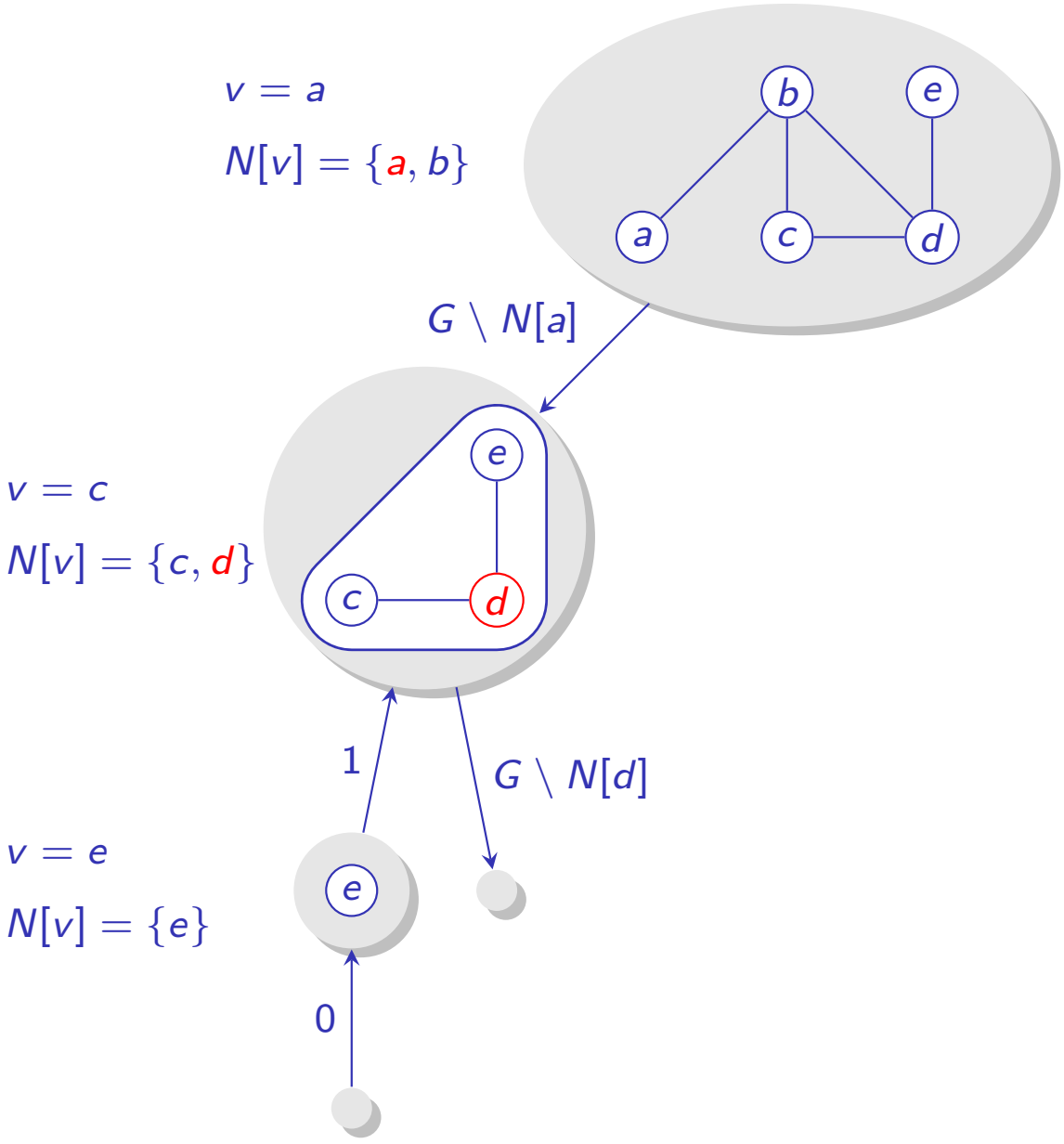

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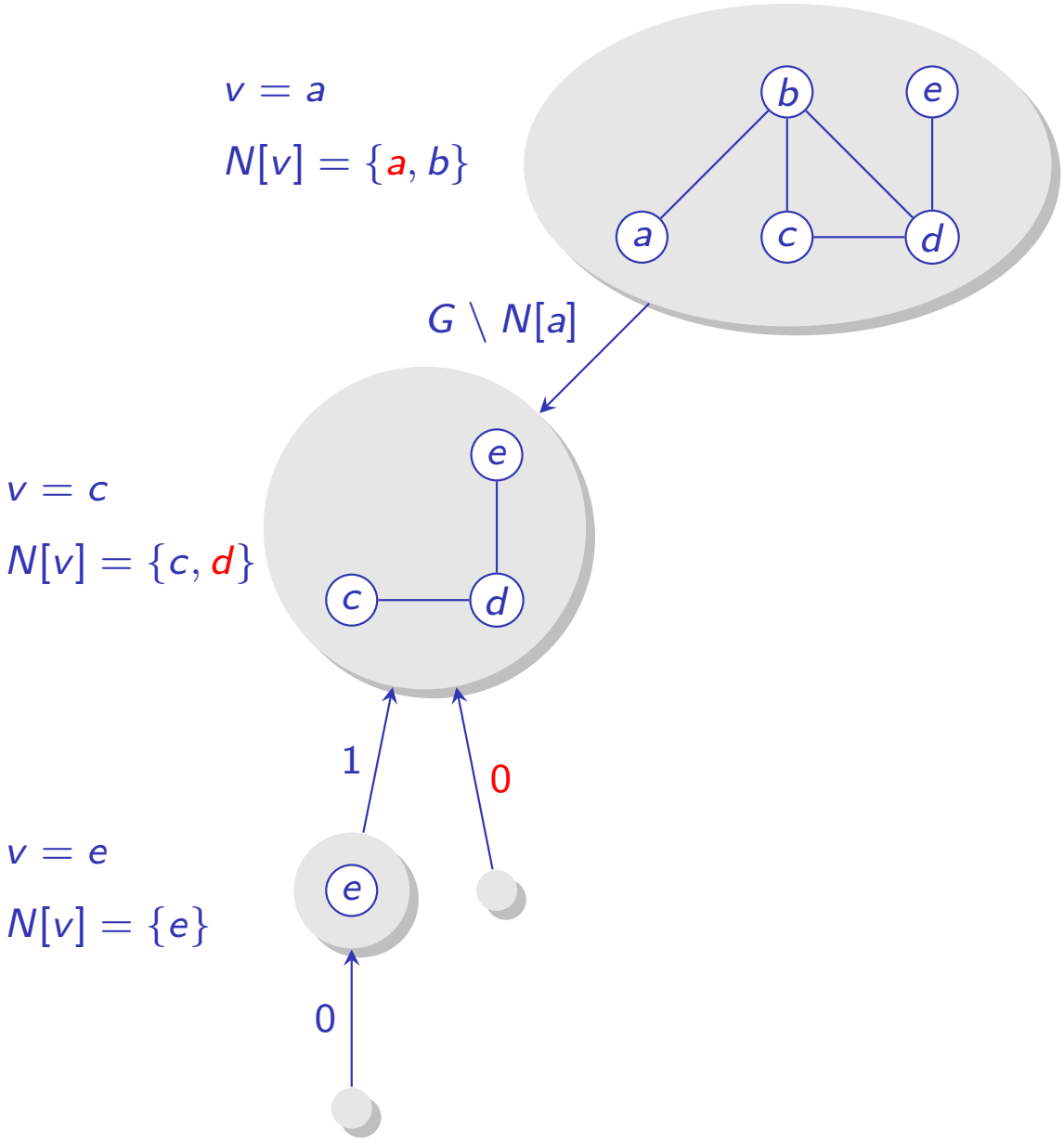

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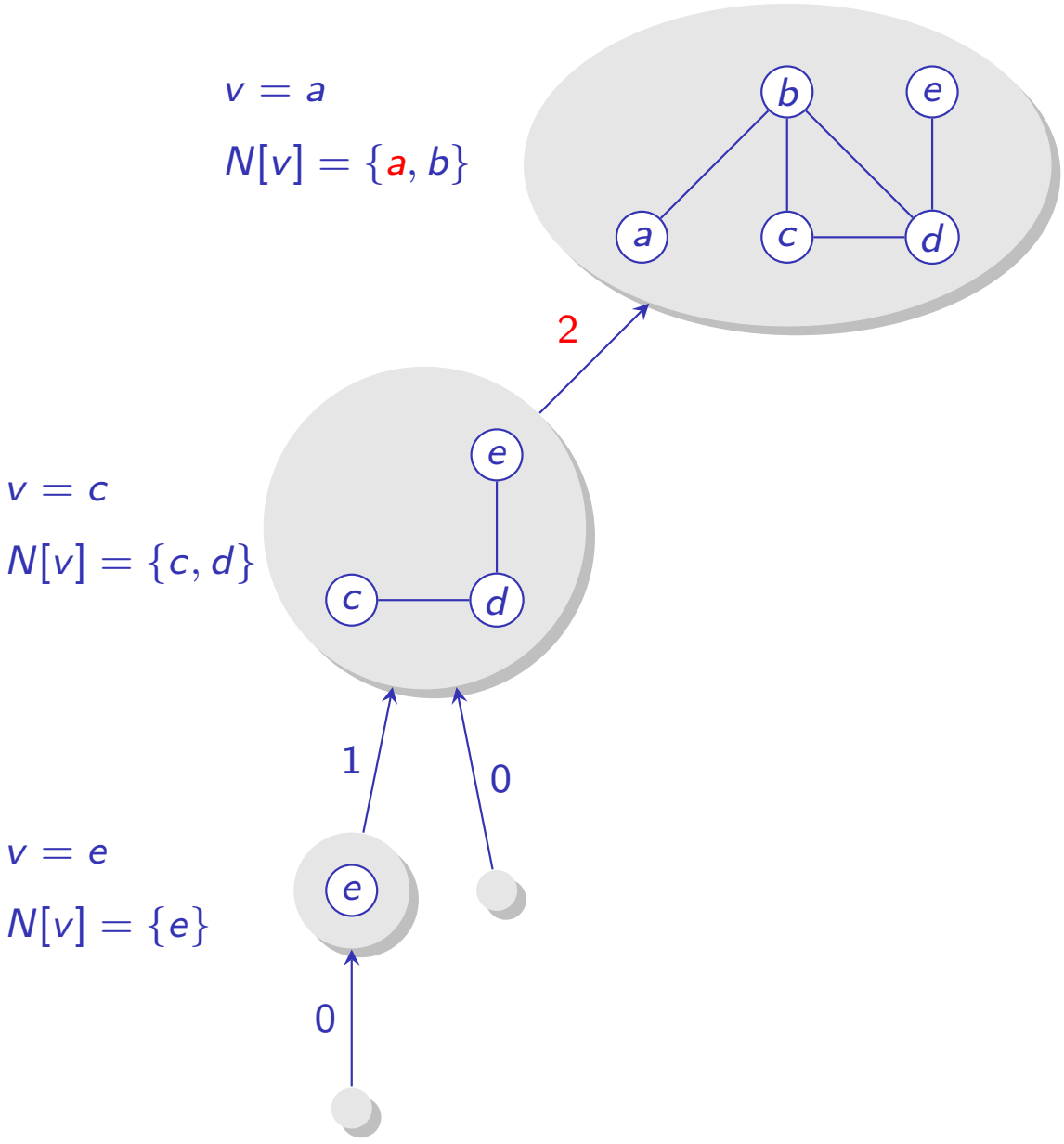

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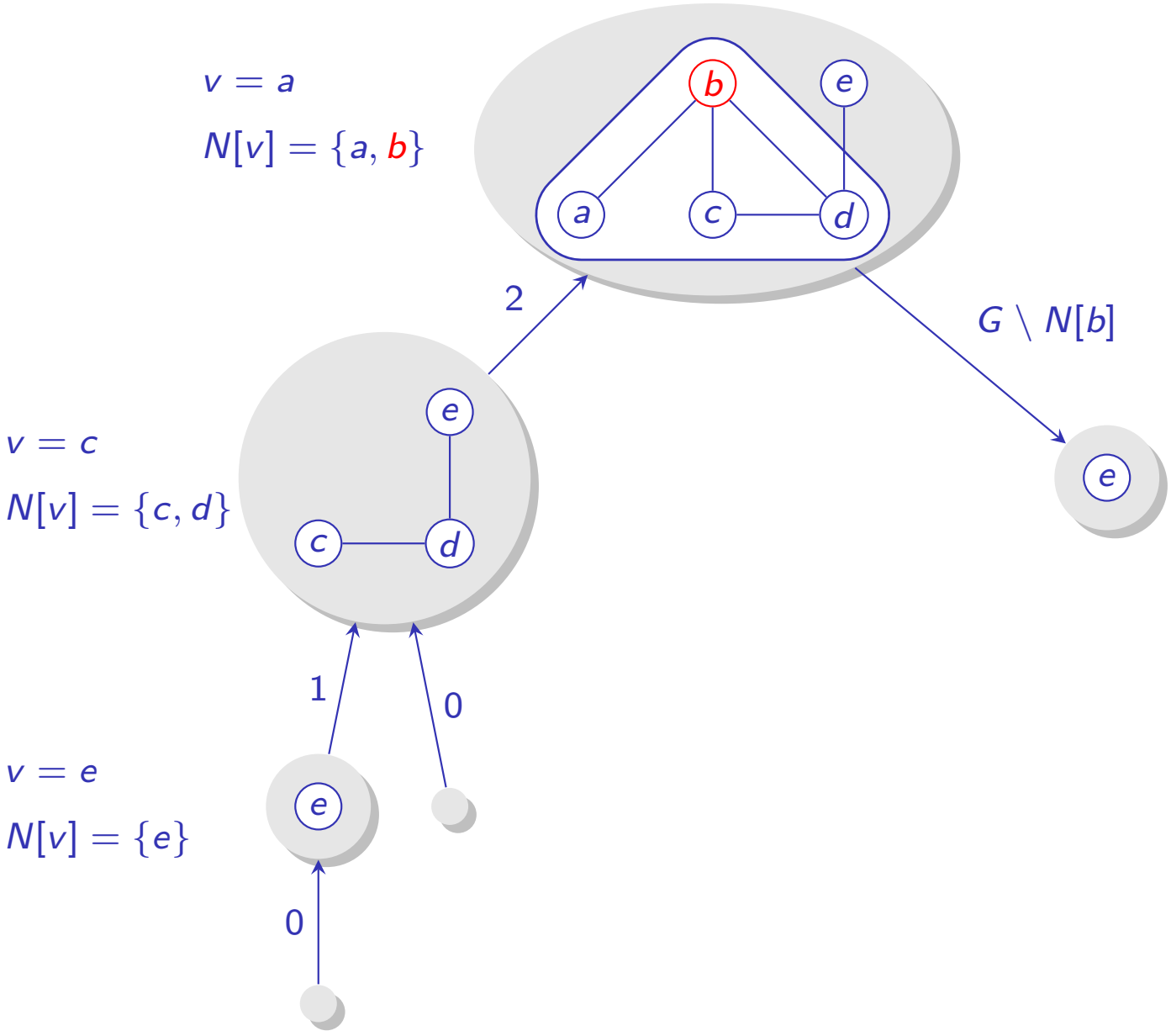

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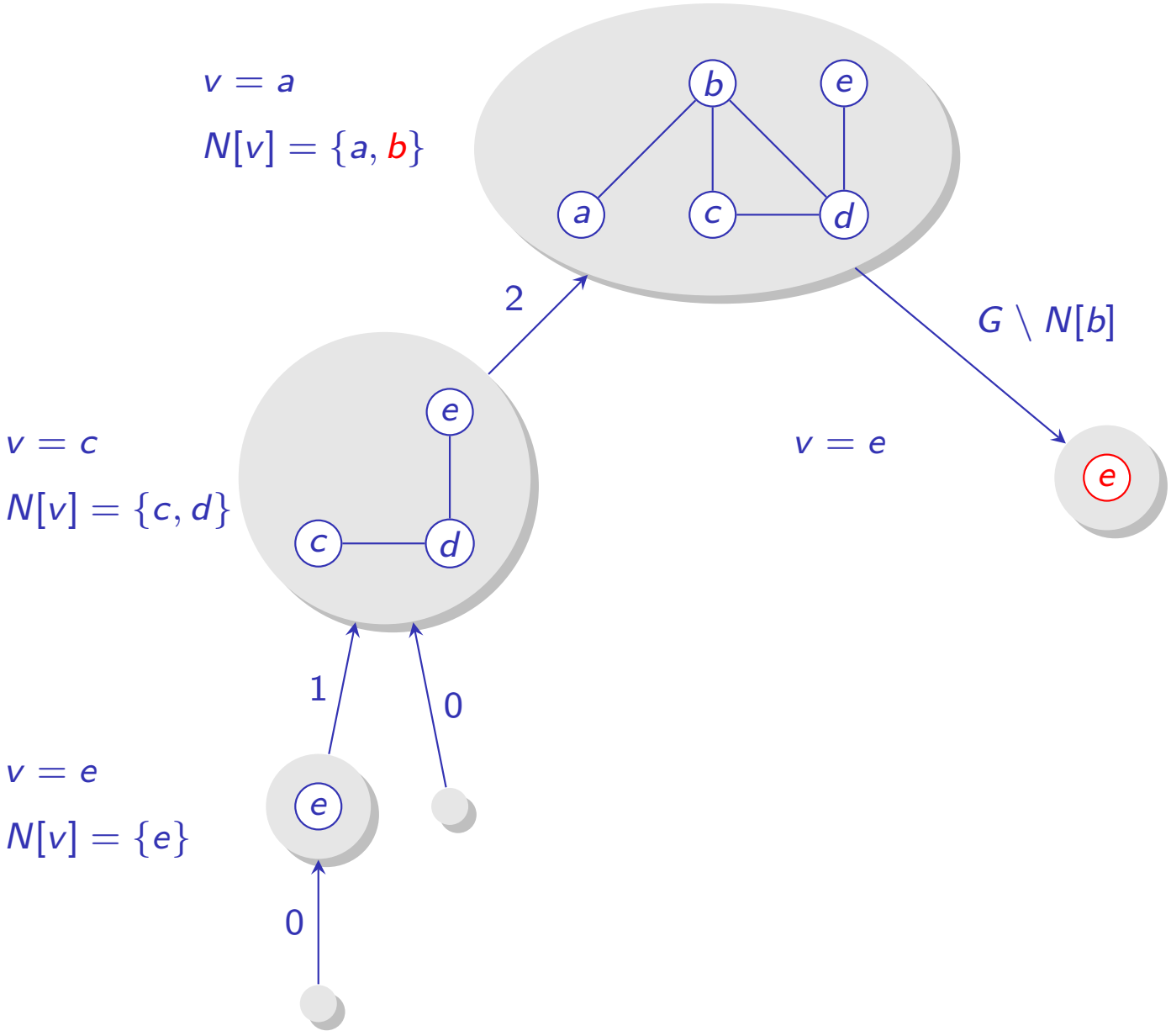

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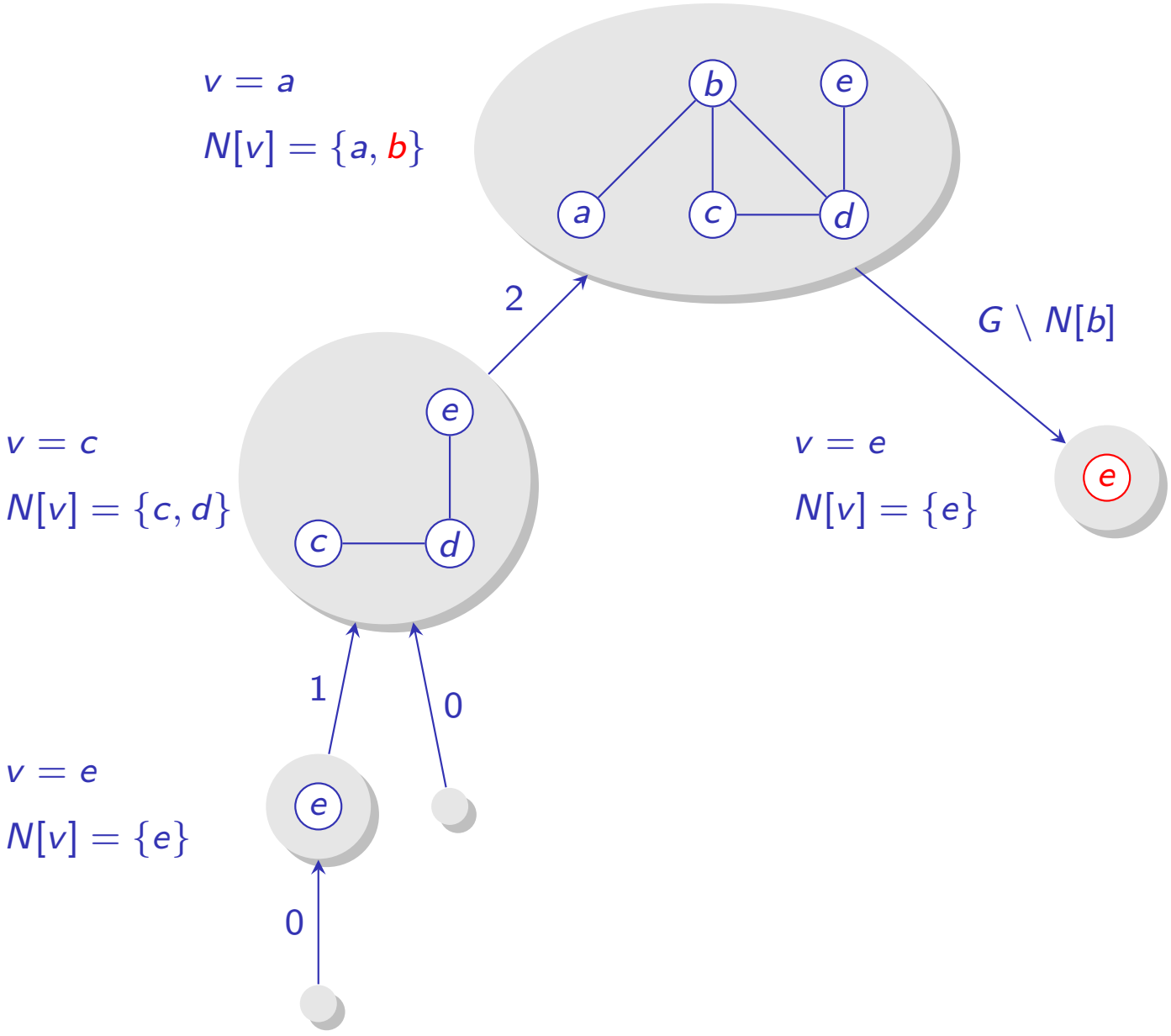

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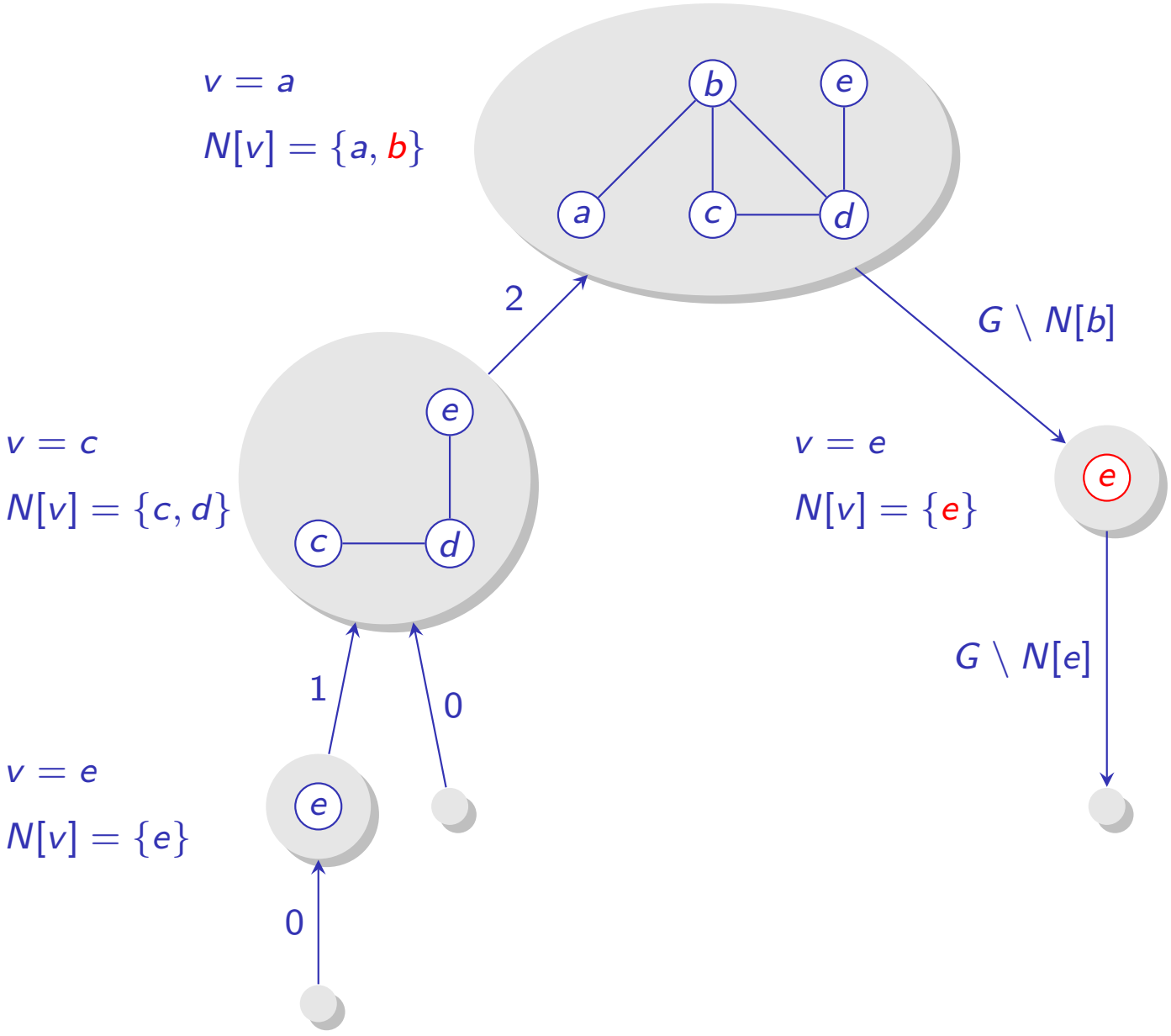

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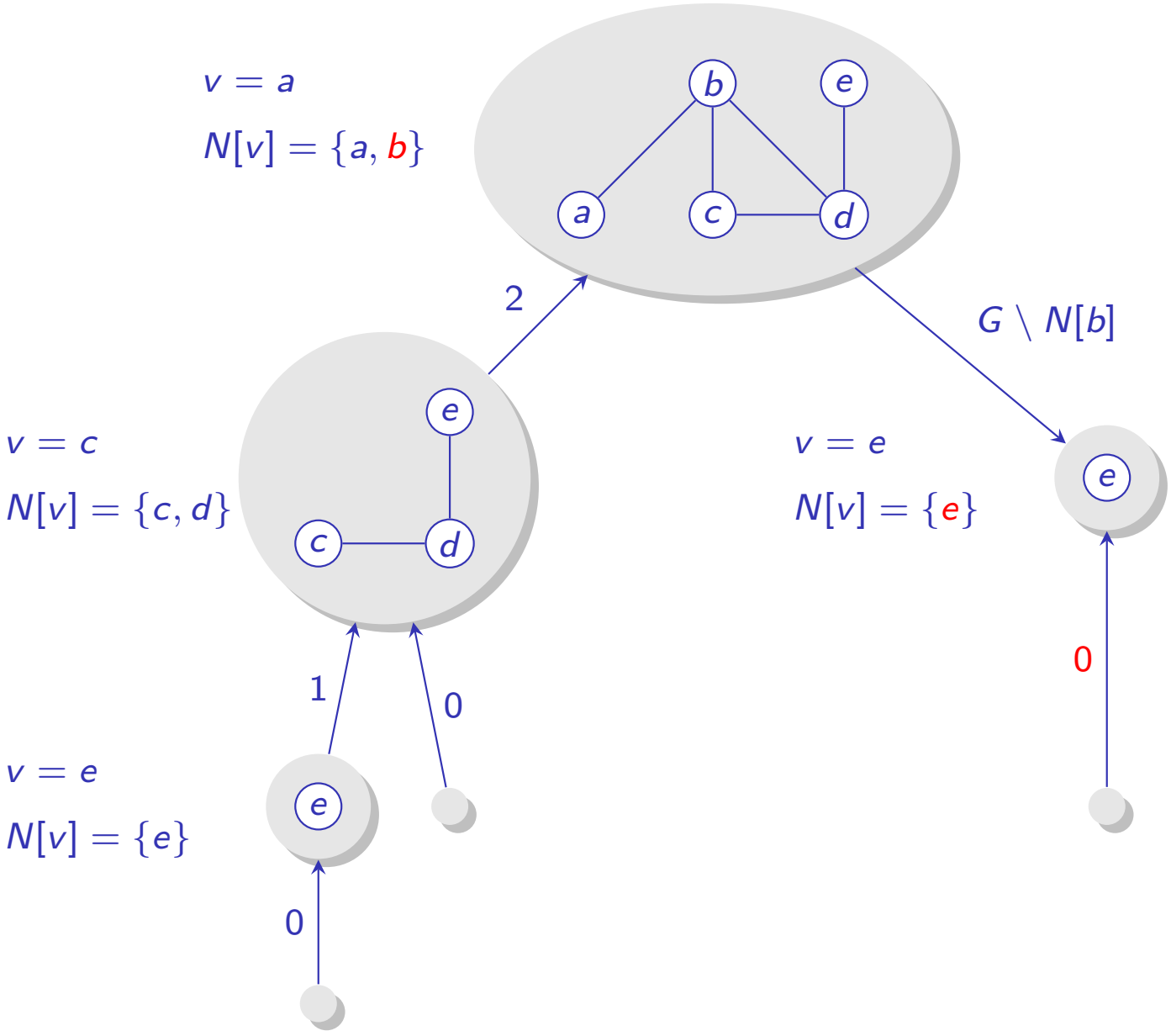

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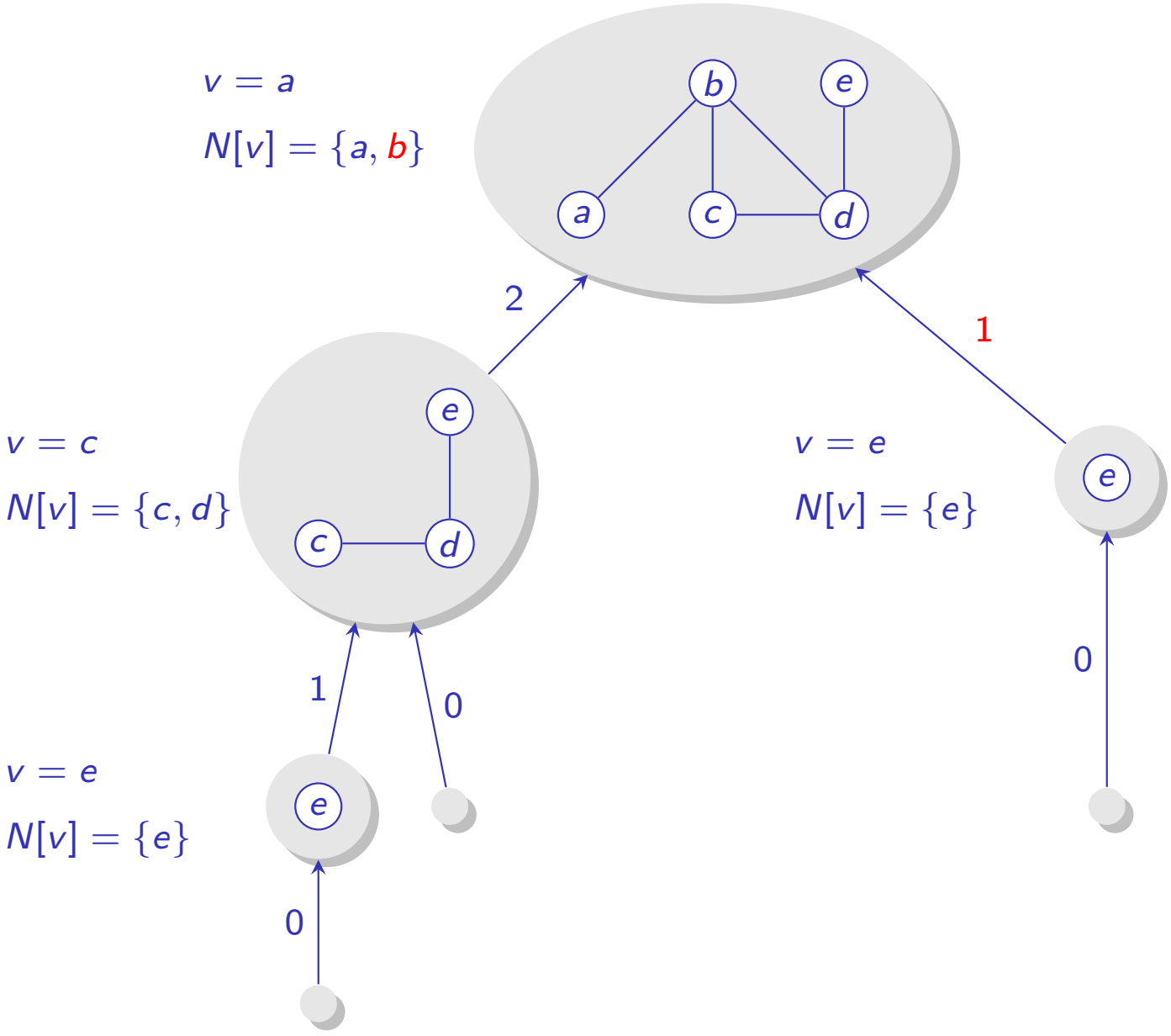

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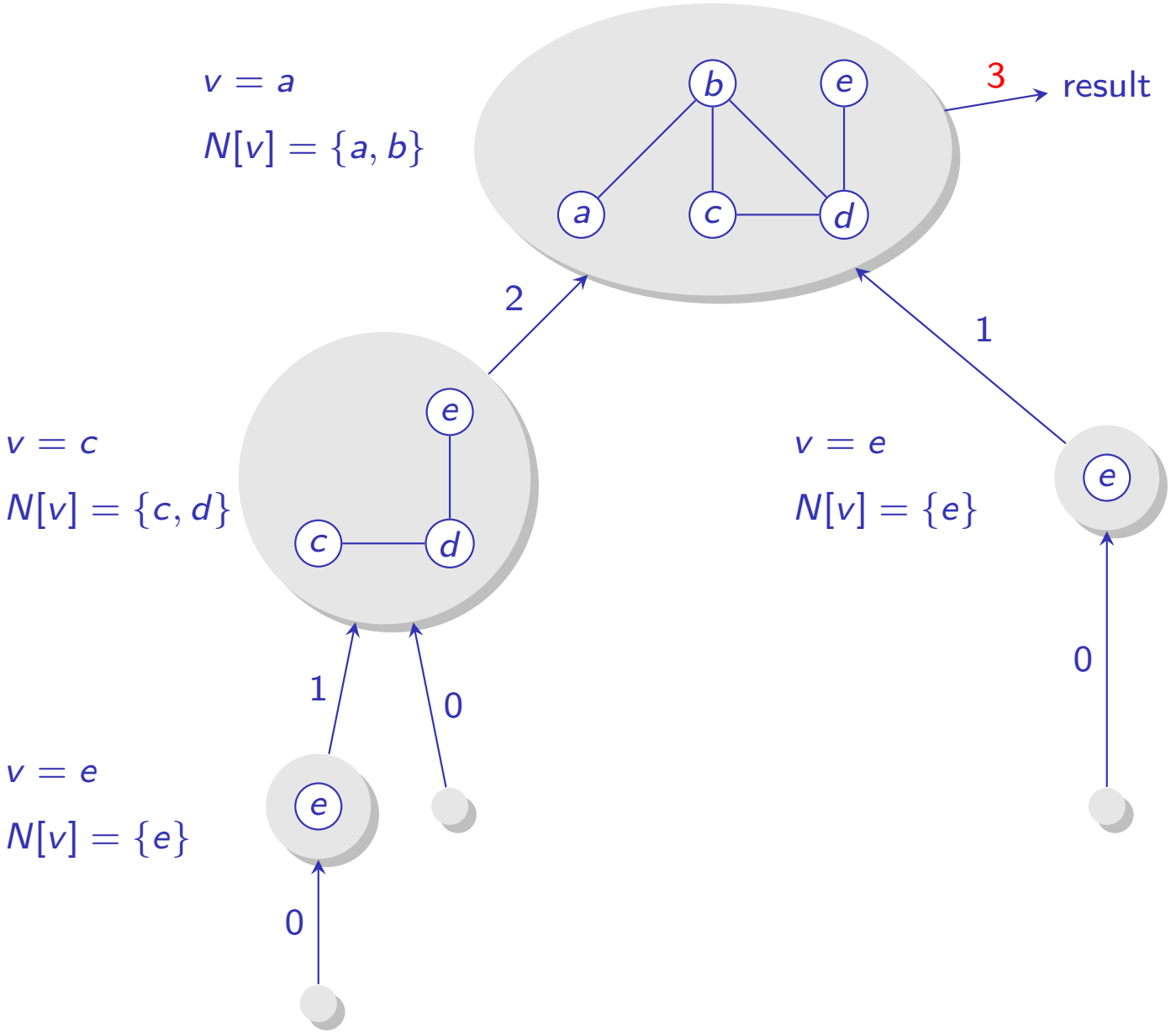

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for some $v \in V$ of minimum degree

$$\leq 1 + (d(v)+1) \cdot T\Big(n - (d(v)+1)\Big)$$

since $T(\cdot)$ is nondecreasing

$$= 1 + s \cdot T(n-s)$$

where $s = d(v)+1$ and thus $s \in \{1, \ldots, n\}$

Lemma

$T(n) \in \mathcal{O}(3^{n/3}) \subset \mathcal{O}(1.44225^n)$

"Proof" (spot the error).

$$T(n) \leq 1 + s \cdot T(n-s)$$

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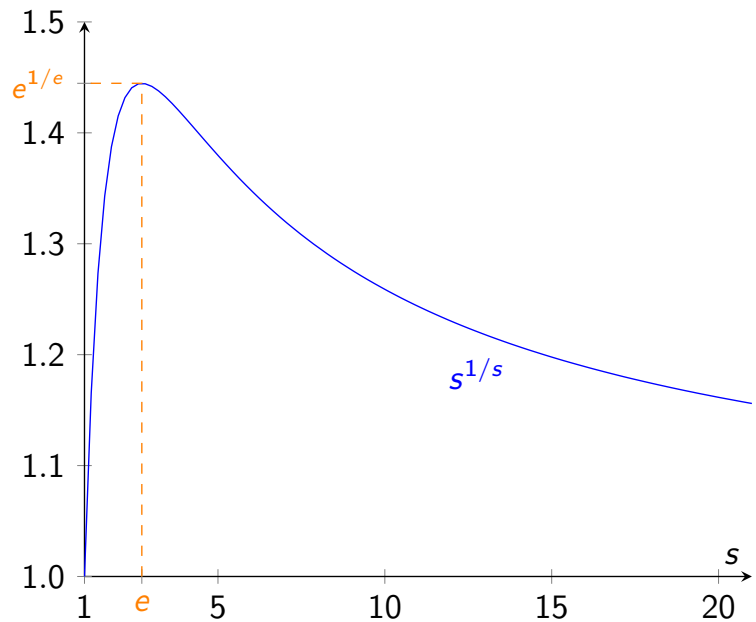

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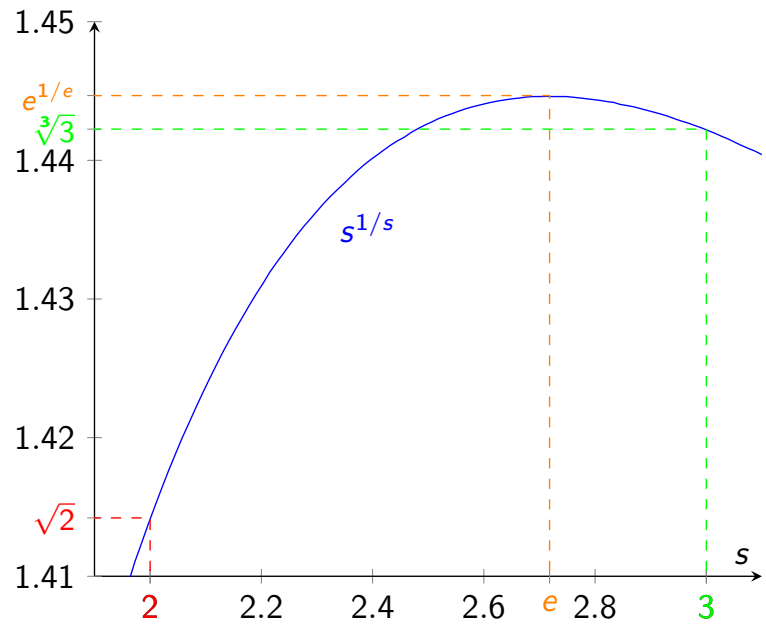

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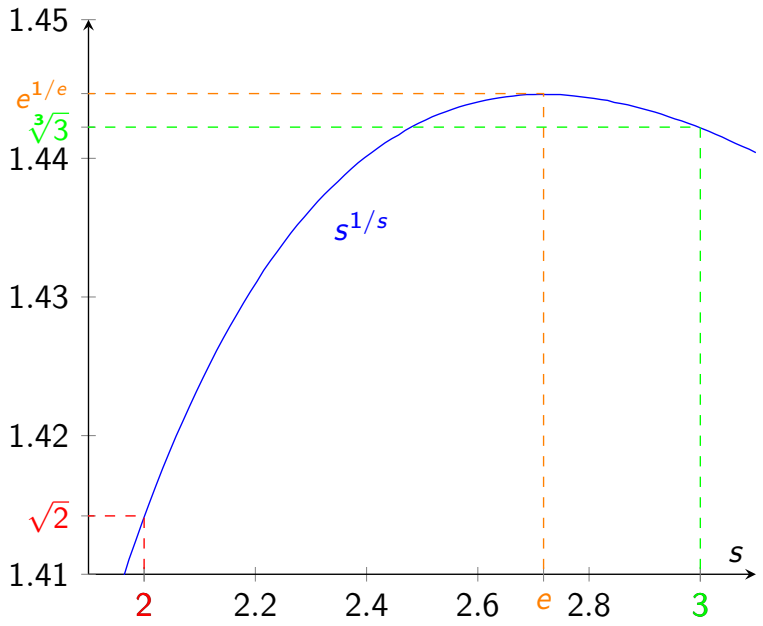

AADS Lecture on EE/FPT, Part 3

Parameterized problems

"Bar fight prevention" aka k-Vertex Cover

Problem: Bouncer in a small city wants to block people at the door to prevent fights. Assume he knows everyone and knows which pairs of people would fight if they were both let in. Management only allows him to block $\leq k$ of the n people who wants in. Is that enough to prevent fights, and if so, who should be blocked?

Equivalent Problem: Given a graph (V, E) with $n = |V|$ vertices, is there a subset $C \subseteq V$ of size $|C| \leq k$ such that every edge has at least one endpoint in C? Such a set C is called a *k-Vertex Cover* in the graph, and its complement $V \setminus C$ is an *Independent Set* of size $n - k$.

For concreteness in the following, suppose $n = 1000$ and $k = 10$.
Naive 1: Try all 2^n subsets of people. ($2^{1000} \approx 1.07 \cdot 10^{301}$ cases).
Naive 2: Use Exact MIS algorithm. ($2 \cdot 3^{1000/3} - 1 \approx 2.195 \cdot 10^{159}$ cases).
Better 1: Try all $\binom{n}{k}$ subsets of k people. ($\binom{1000}{10} \approx 2.63 \cdot 10^{23}$ cases).

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"Bar fight prevention" via Kernelization

Consider the conflict graph $G = (V, E)$.

Idea: If $d(v) = 0$: let v in and drop v from G.
Why?

Idea: If $d(v) > k$: reject v, drop v from G, and decrease k.
Why?

Note: If $d(v) \leq k$ for all v and $|E| > k^2$, there is no solution.
Why?

Better 2: The above ideas reduce to a graph H with $|V| \leq 2k^2$ vertices.
Why?

Now try all $\binom{2k^2}{k}$ subsets of k people. $\left(\binom{2 \cdot 10^2}{10} \approx 2.24 \cdot 10^{16}\right)$.

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Consider the conflict graph $G = (V, E)$.

Idea: If $d(v) = 0$: let v in and drop v from G.
Why? Safe because no conflicts.

Idea: If $d(v) > k$: reject v, drop v from G, and decrease k.
Why? Not rejecting v means rejecting $d(v) > k$ people.

Note: If $d(v) \leq k$ for all v and $|E| > k^2$, there is no solution.
Why?

Better 2: The above ideas reduce to a graph H with $|V| \leq 2k^2$ vertices.
Why?

Now try all $\binom{2k^2}{k}$ subsets of k people. $(\binom{2 \cdot 10^2}{10} \approx 2.24 \cdot 10^{16})$.

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"Bar fight prevention" via Kernelization

```
1: function BarFightPrevention(k, G)
2:     k', H, C ← BFP-Kernel(k, G)
3:     if H has ≤ (k')^2 edges and BFP-Brute-Force(k', H) returns a solution C' then
4:         return C ∪ C'
5:     return "No solution"

6: function BFP-Kernel(k, G)
7:     k' ← k, H ← G, C ← ∅
8:     loop
9:         if Some v has d(v) = 0 then
10:            H ← H \ {v}
11:        elseif k' > 0 and some v has d(v) > k' then
12:            H ← H \ {v}, C ← C ∪ {v}, k' ← k' − 1
13:        elseif k' > 0 and some v has N[v] = {v, w} for some w then
14:            H ← H \ N[v], C ← C ∪ {w}, k' ← k' − 1
15:        else
16:            return k', H, C

17: function BFP-Brute-Force(k, G = (V, E))
18:     for every subset C ⊆ V of size k do
19:         if C is a vertex cover of G then
20:             return C
21:     return "No solution"
```

Kernelization

The subgraph H we reduced to before brute-forcing is called a *Kernel* for the Bar Fight Prevention problem, and the process of finding such a kernel is called *Kernelization*.

The general idea is to use the parameter k to quickly reduce to a smaller problem, whose size ideally depends only on k and not on n.

For the bar fight prevention problem we have just shown that:

- If there is a solution for a given k and a given graph G with n vertices and m edges, then we can find a kernel H with at most k^2 vertices.
- Furthermore, such a kernel can be found in $\mathcal{O}(m+n)$ time, and checking if a given subset of size at most k is a solution can be done in $\mathcal{O}(k^2)$ time.
- Thus, for any fixed k, the total running time of this algorithm is $\mathcal{O}(m+n+\binom{k^2}{k}k^2) \subseteq \mathcal{O}(m+n+k^{2k+2}) \subseteq \mathcal{O}_k(m+n)$.

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"Bar fight prevention" via Bounded Search Tree

Note: For each edge $(u, v) \in E$, at least one of u, v must be rejected.

Idea: Pick arbitrary edge (u, v), and recursively try with u rejected and with v rejected.

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1: function BFP-Bounded-Search(k, G)
2:     if G has no edges then
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4:     if k > 0 then
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6:         for w ∈ {u, v} do
7:             if BFP-Bounded-Search(k − 1, G \ {w}) returns a solution C then
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This recursive procedure has depth at most k.
Thus the total number of subproblems considered at most 2^k.
If we start by rejecting all vertices of degree $d(v) > k$ (like in the kernelization approach), the resulting graph has at most $|E| = \frac{1}{2}\sum_{v \in V} d(v) \leq \frac{1}{2}nk$ edges, so constructing each subproblem can be done in $\mathcal{O}(nk)$ time.
The total running time is then $\mathcal{O}(m + nk \cdot 2^k)$. $(1000 \cdot 10 \cdot 2^{10} \approx 10^7)$
Part of Assignment 5 asks you to improve this.

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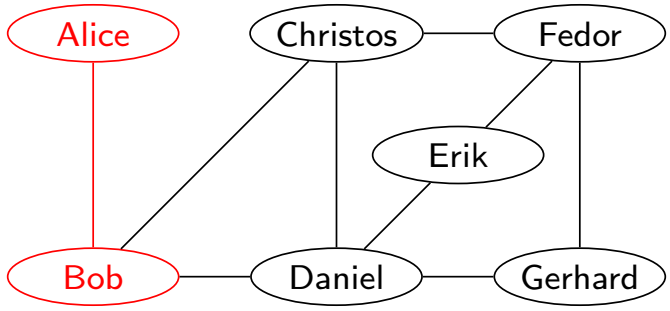
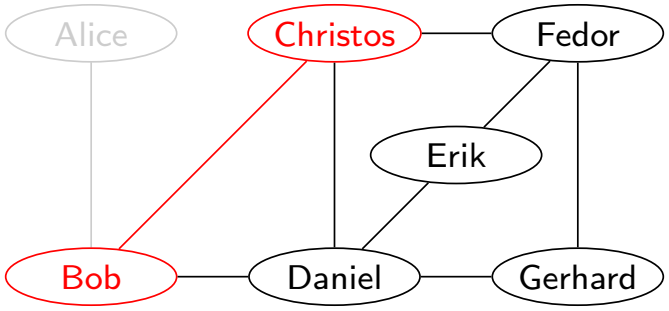


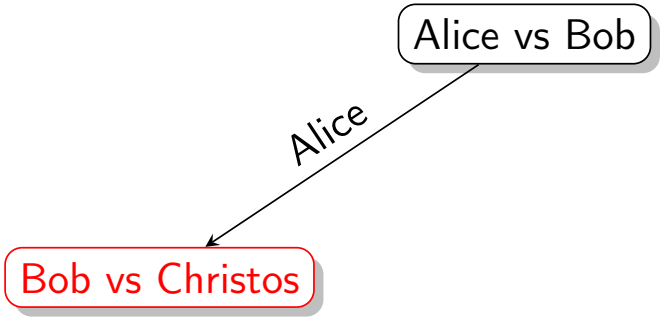
$k = 3$

Alice vs Bob

“Bar fight prevention” via Bounded Search Tree



$k = 2$



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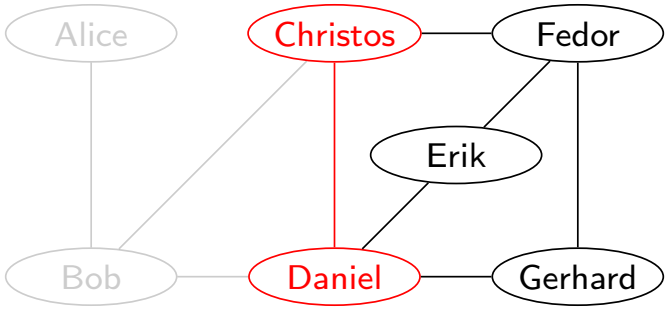
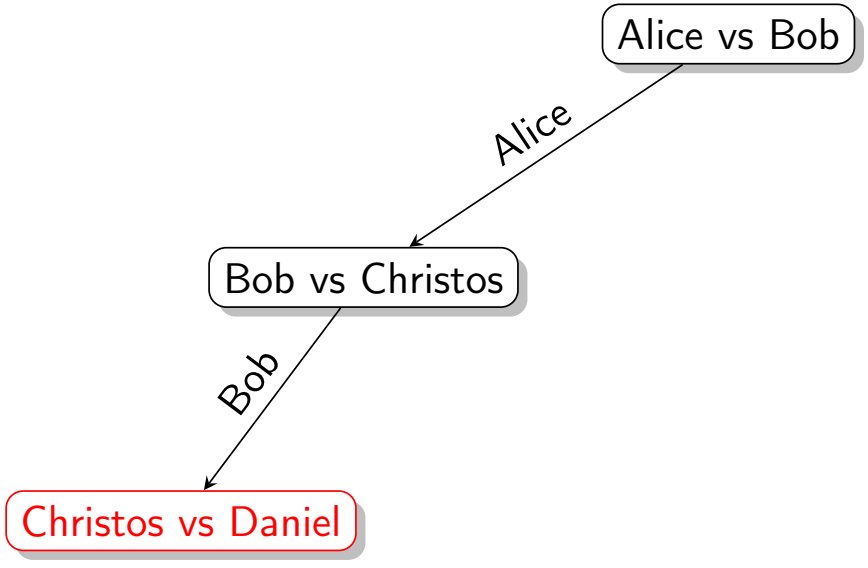


$k = 1$



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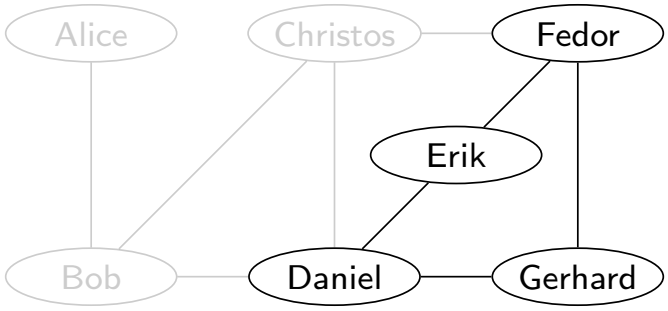

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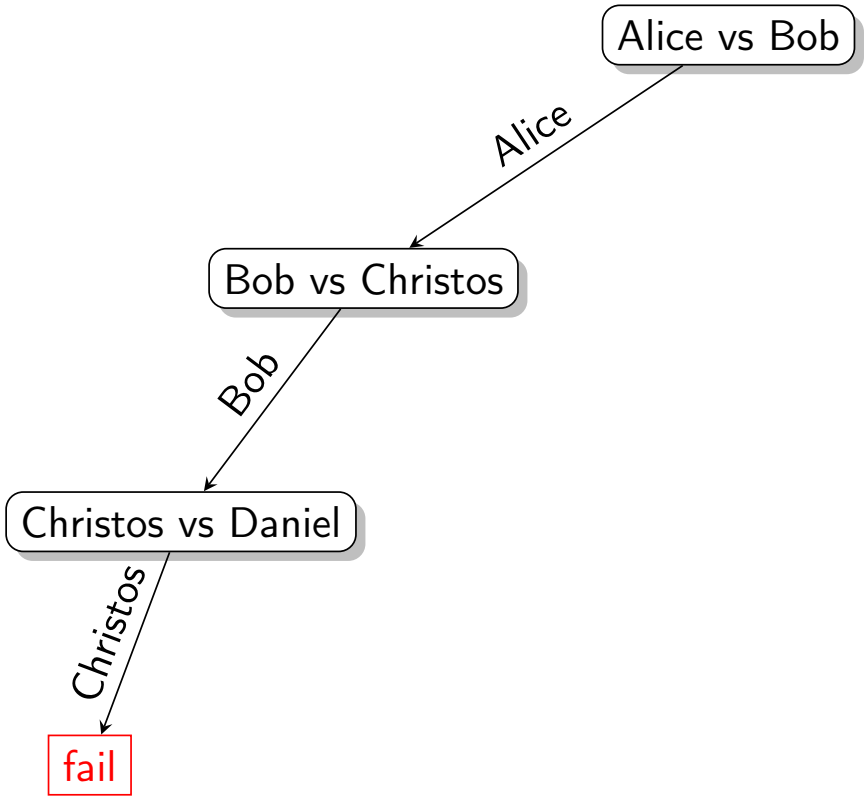

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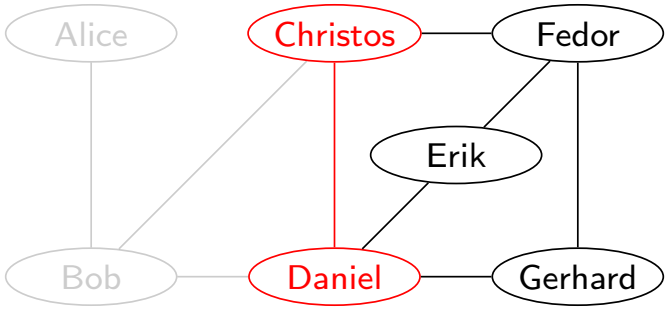

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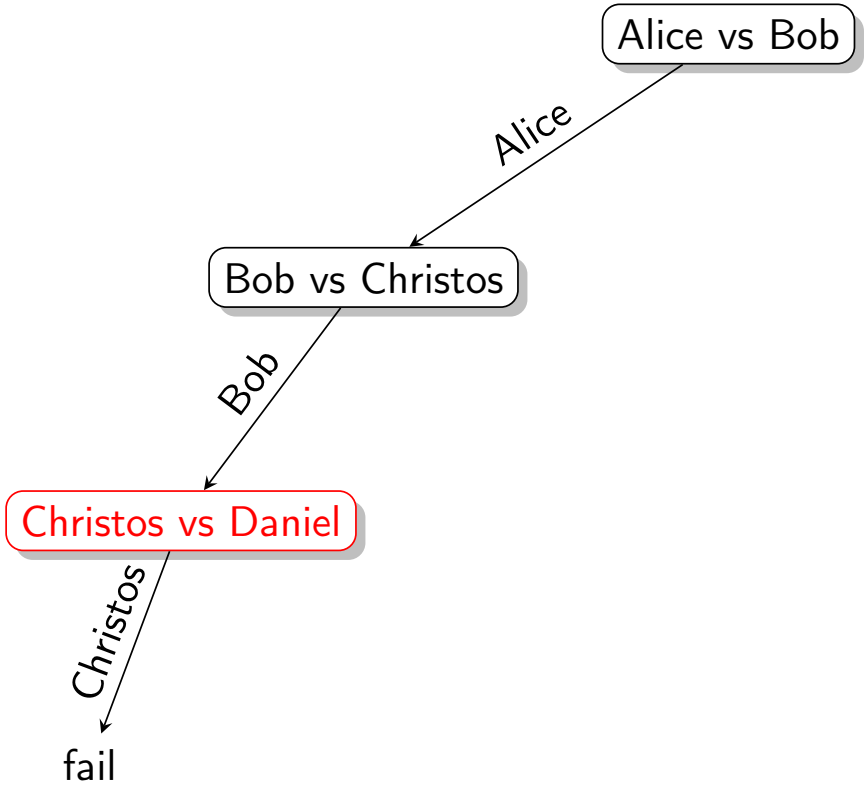

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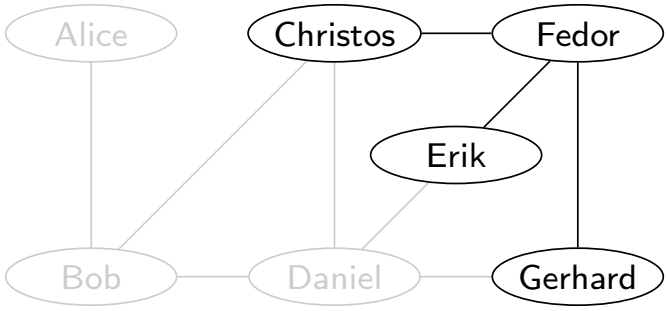

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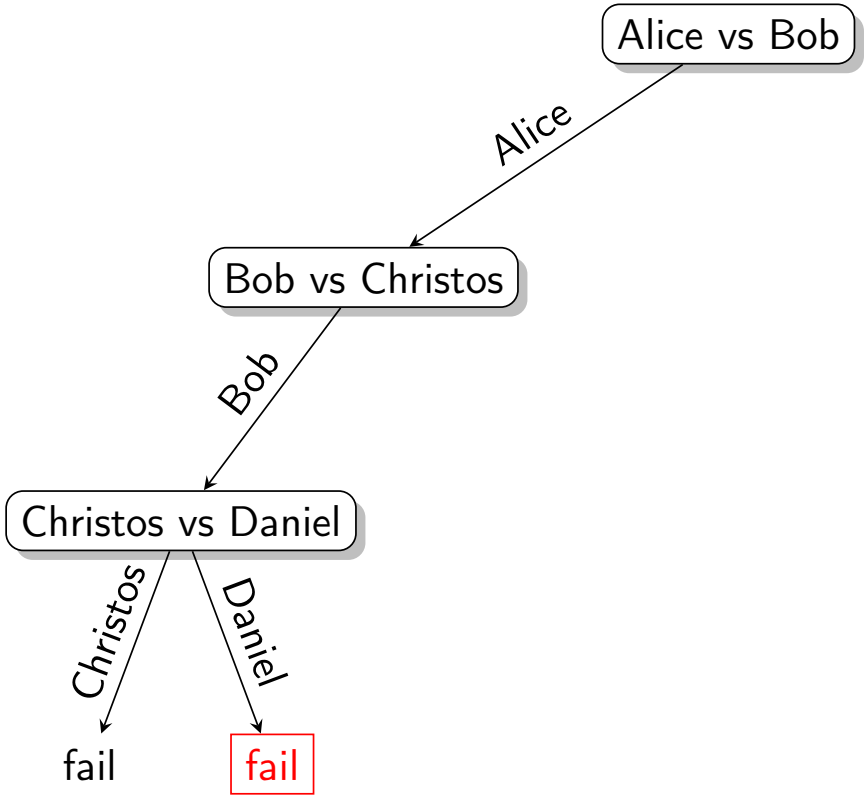

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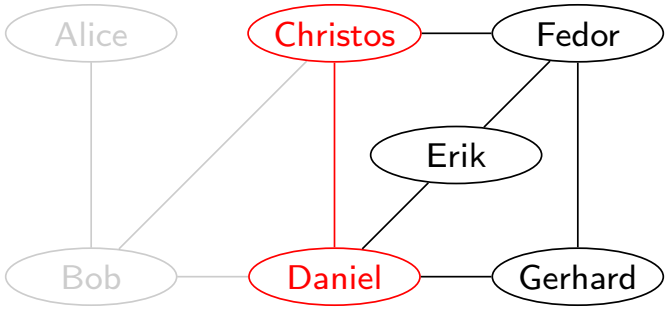

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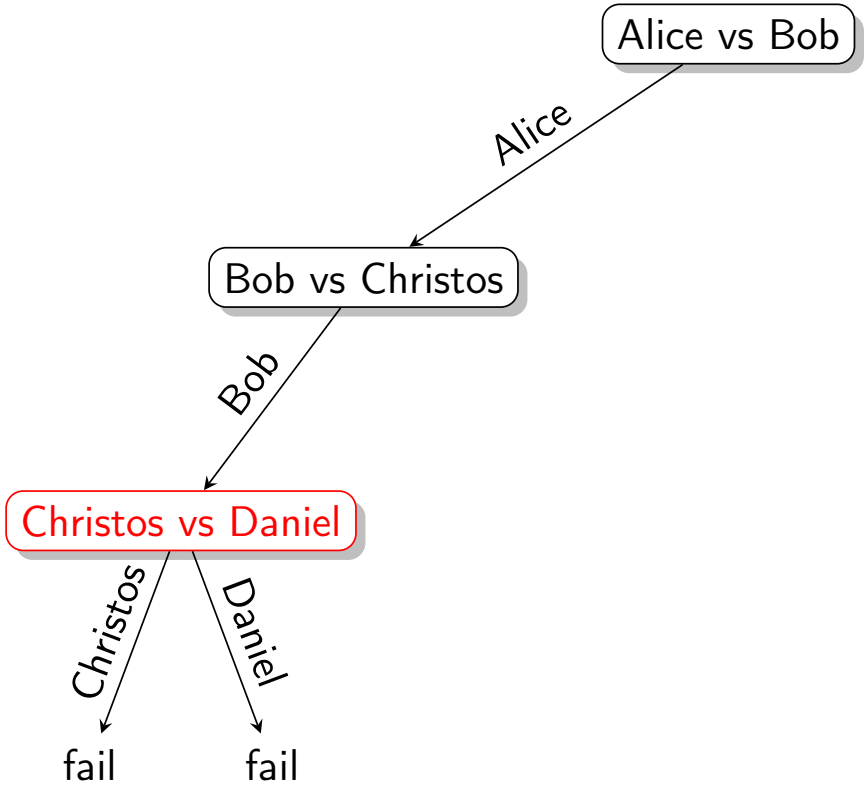

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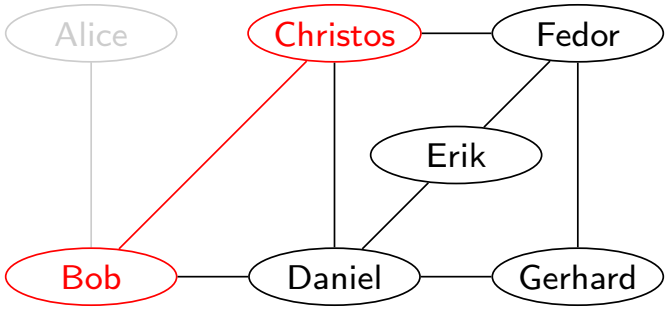

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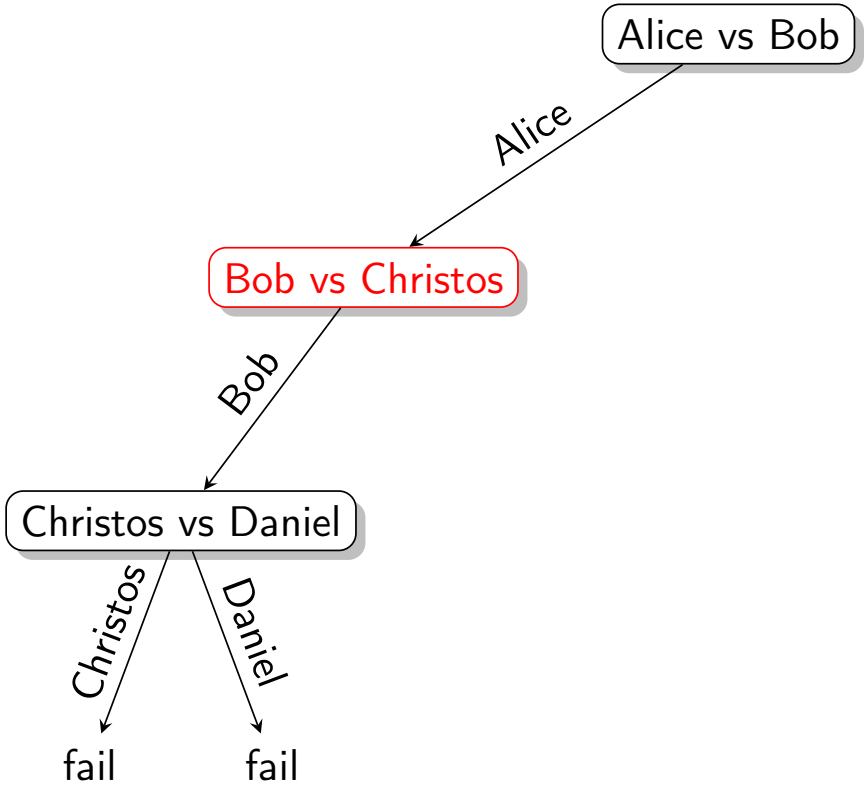

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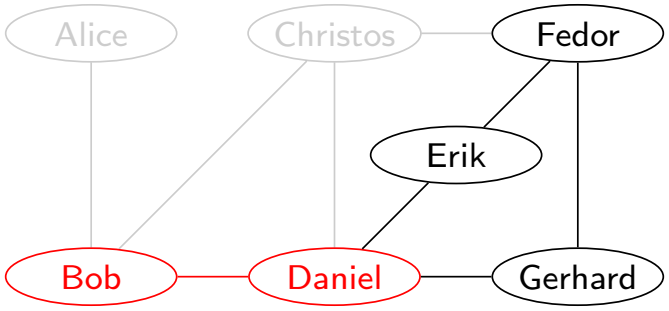

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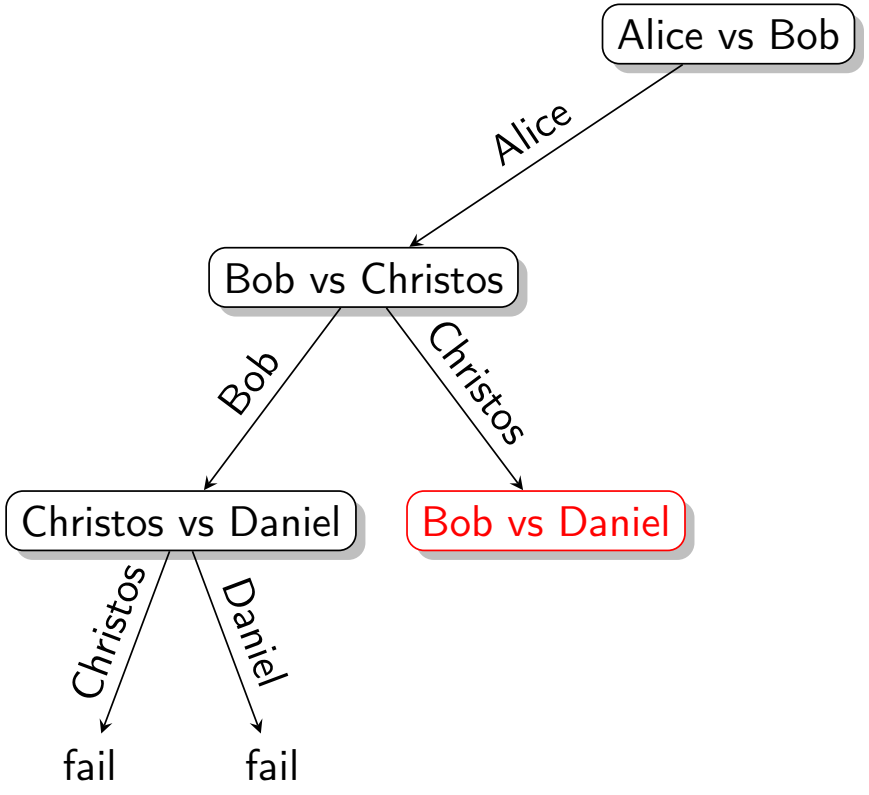

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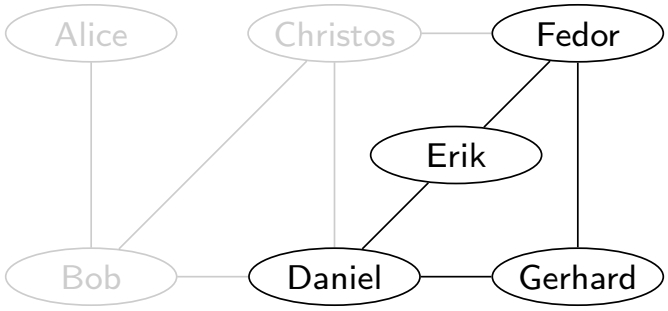

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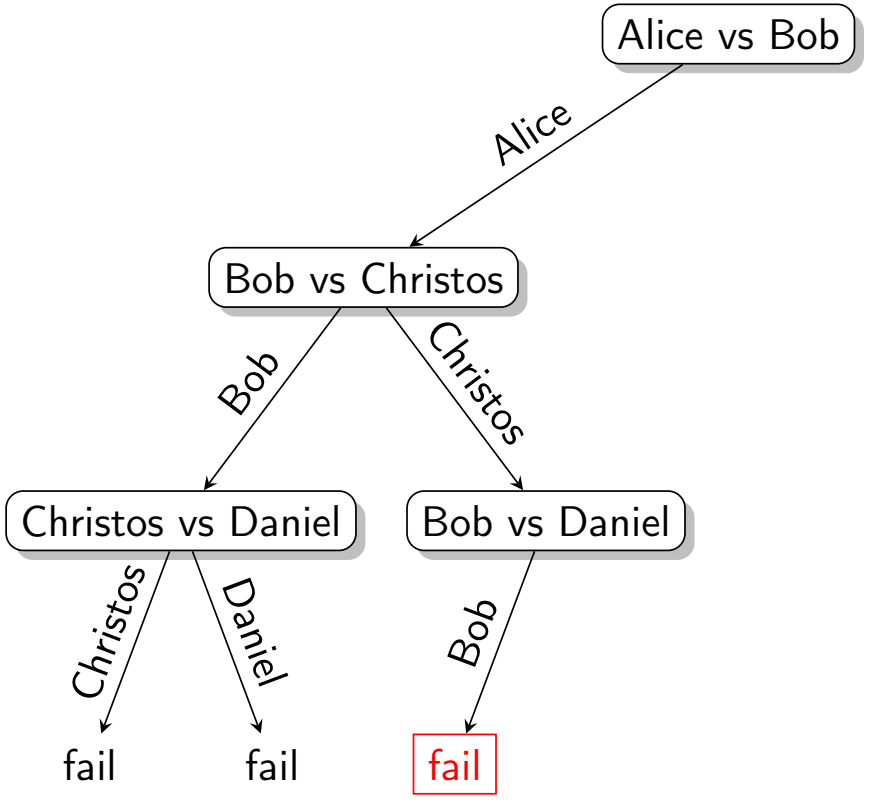

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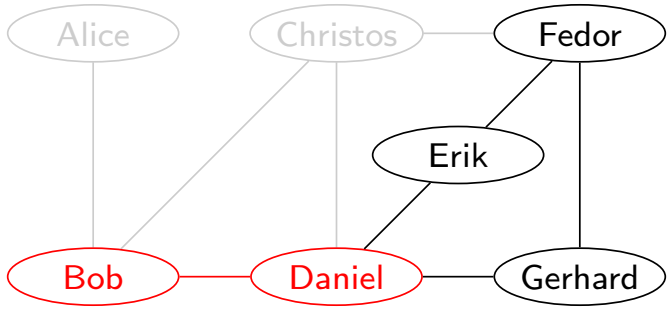

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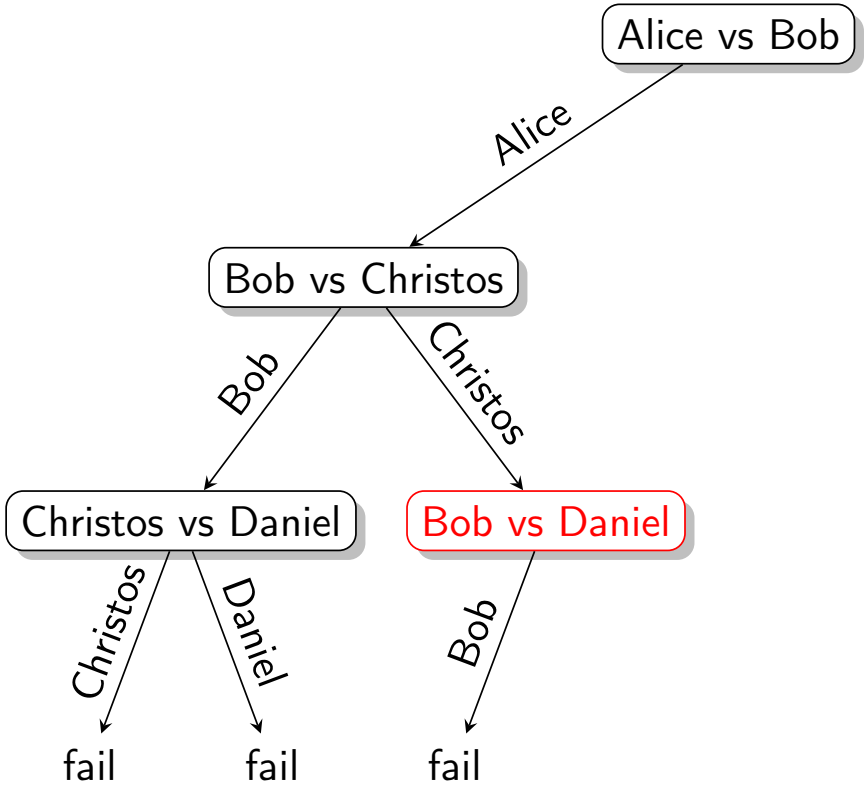

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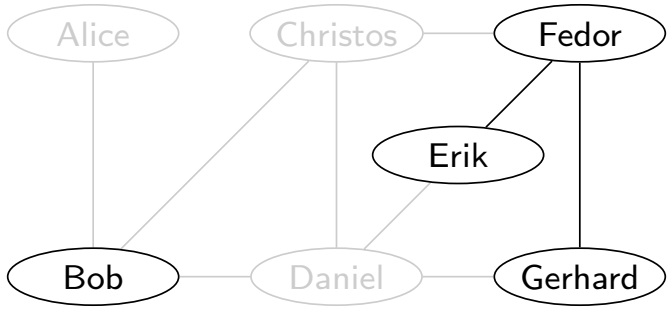

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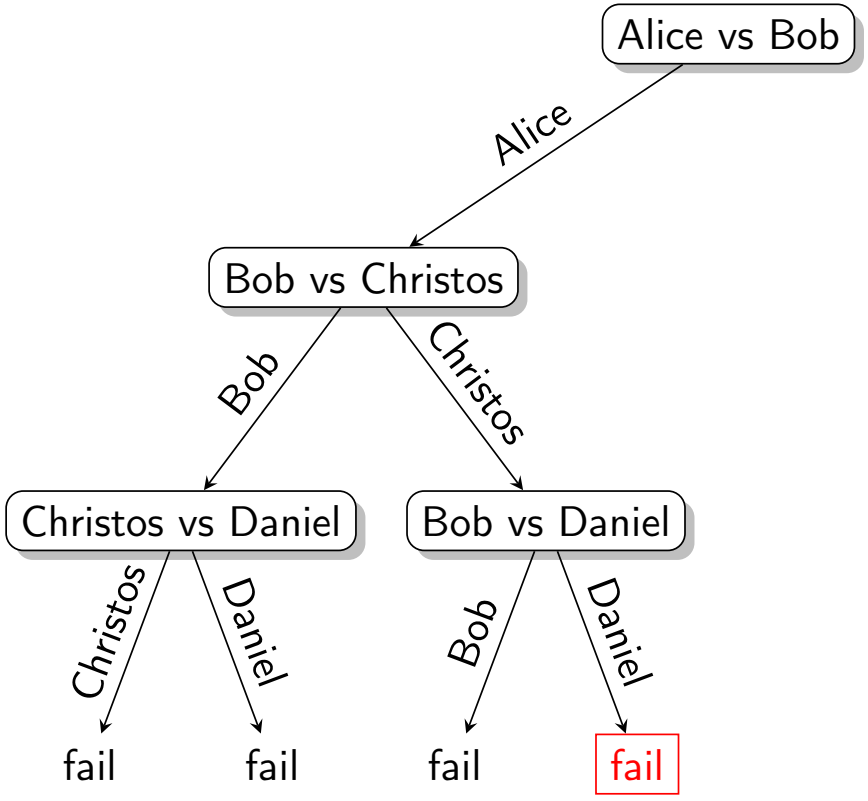

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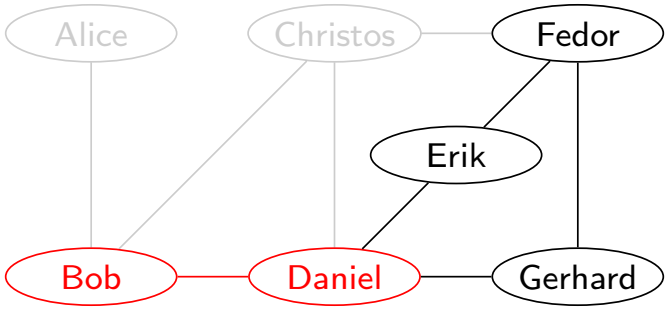

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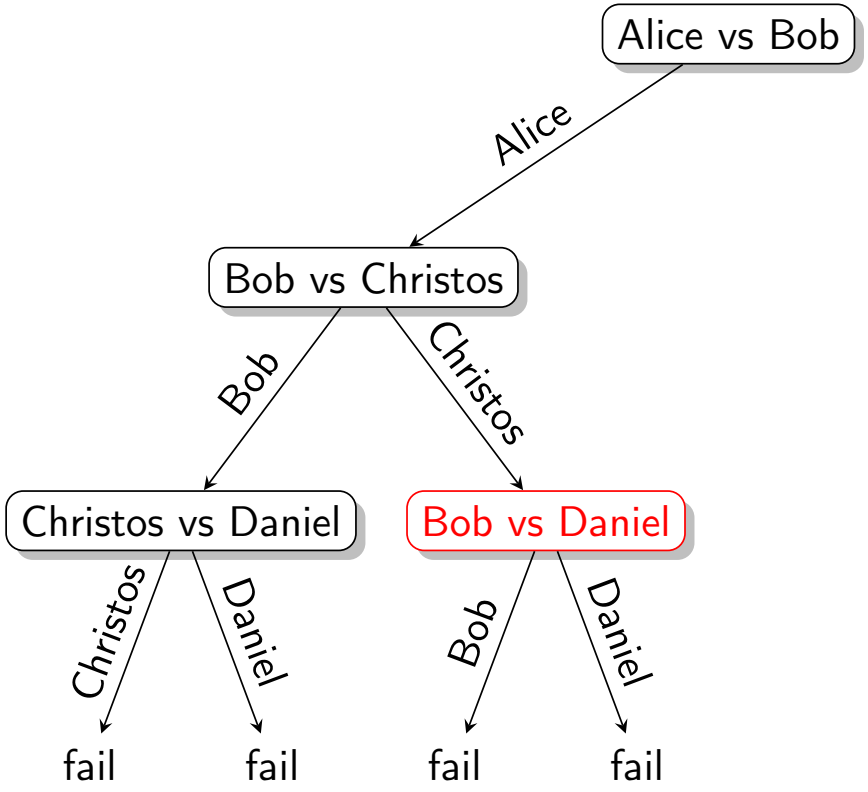

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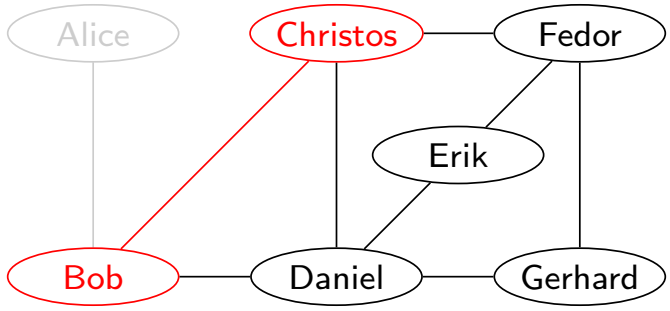

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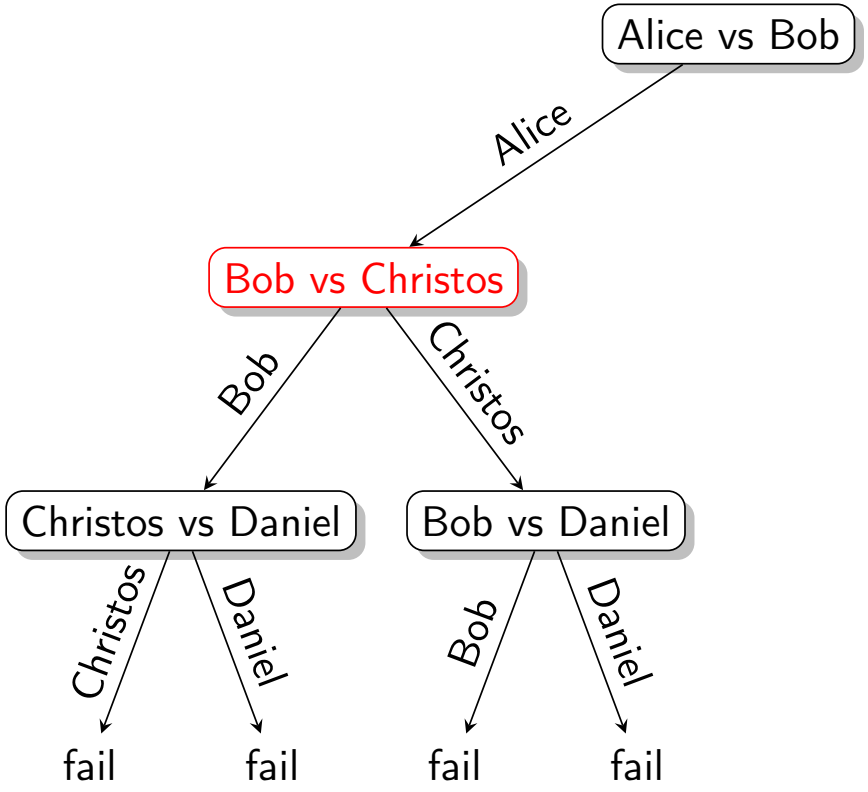

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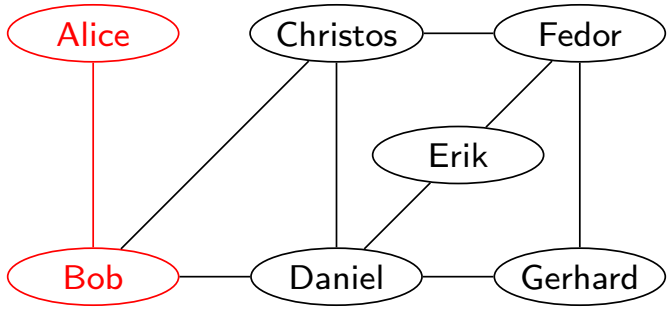

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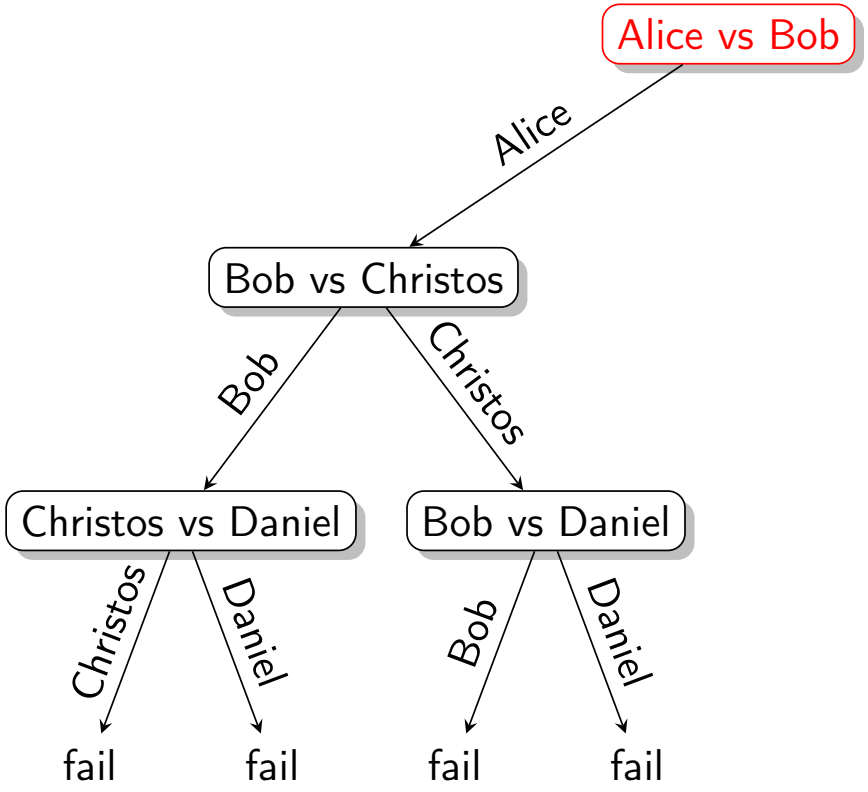

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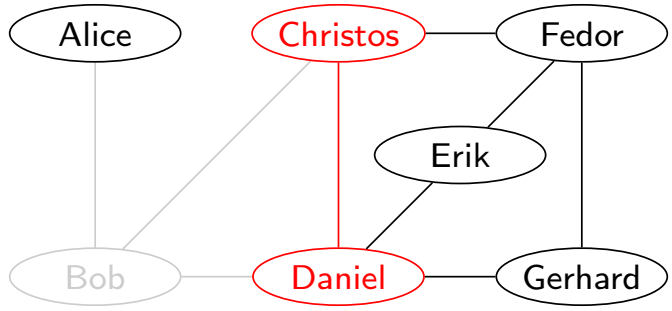

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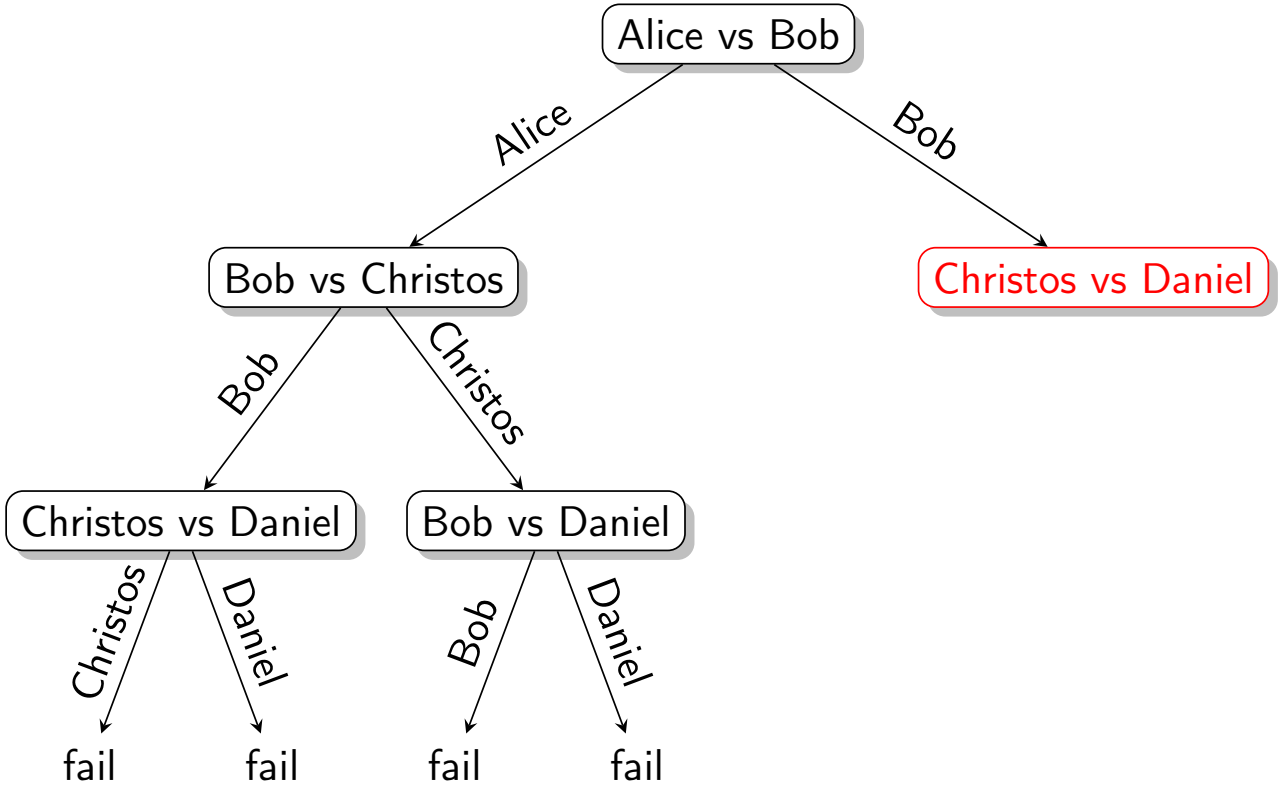

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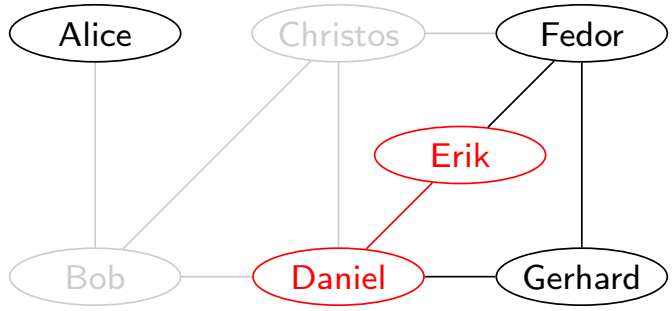

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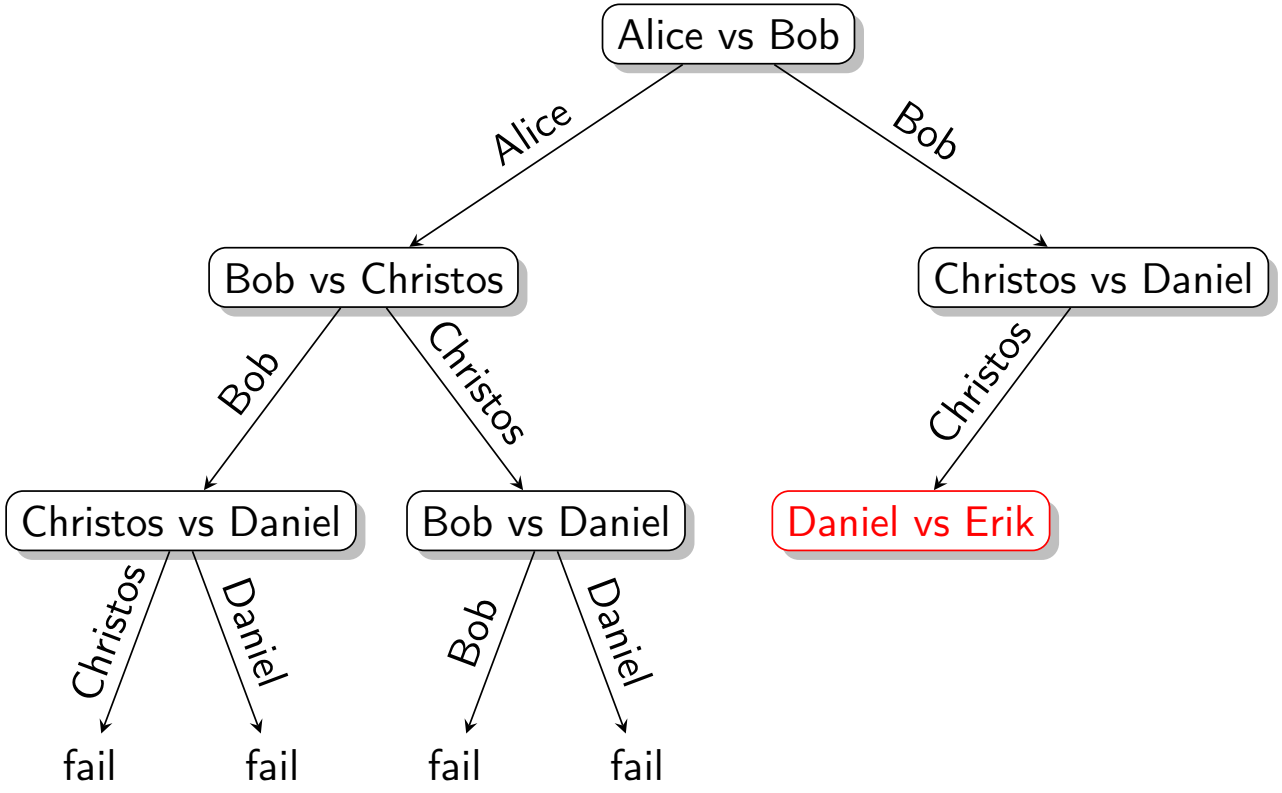

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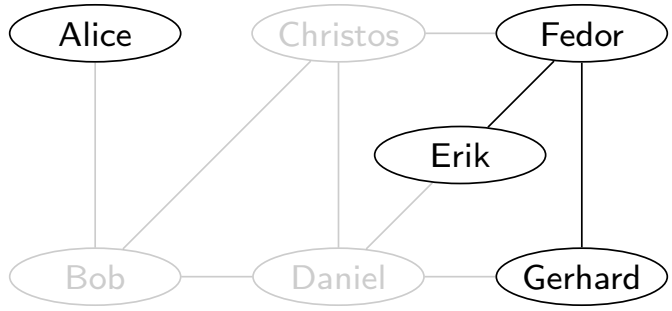

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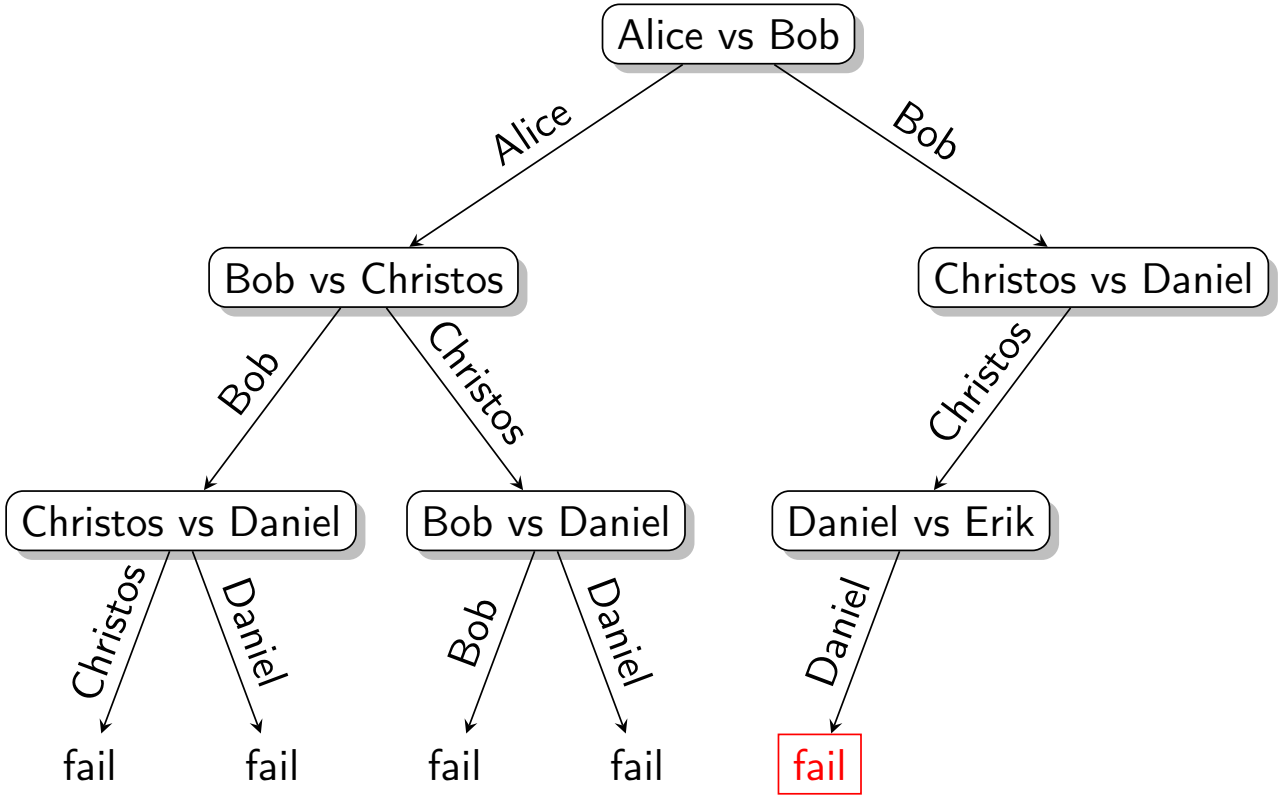

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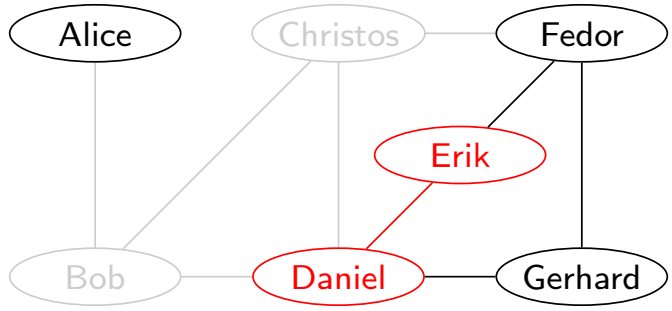

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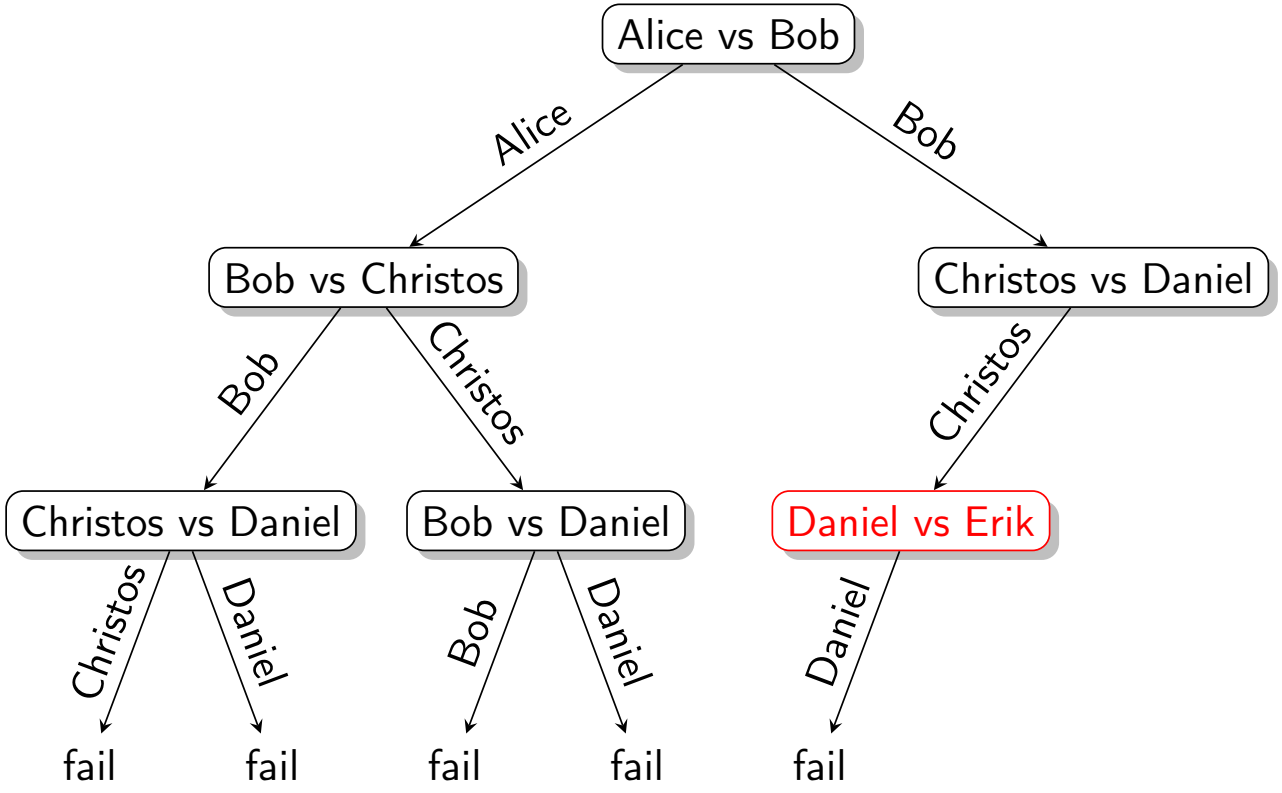

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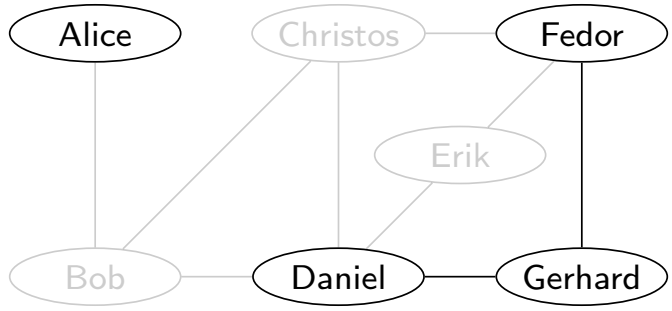

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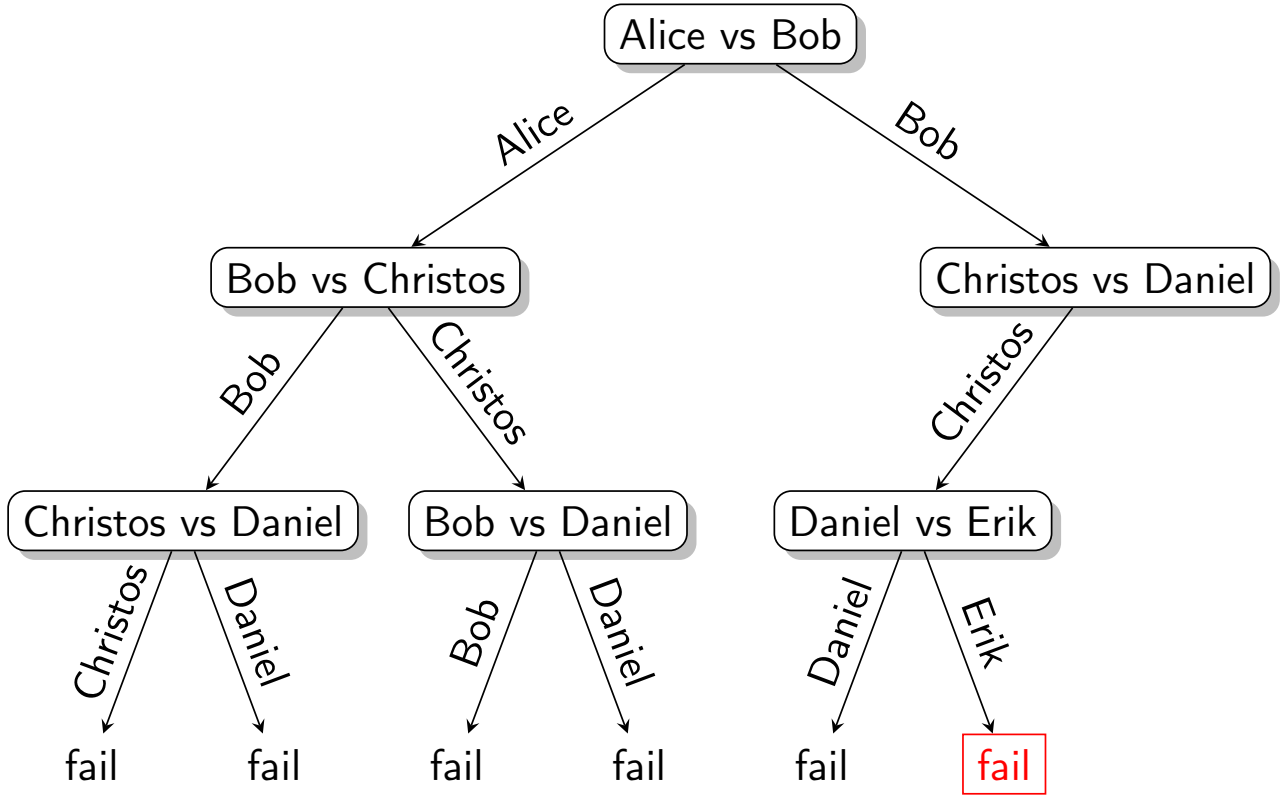

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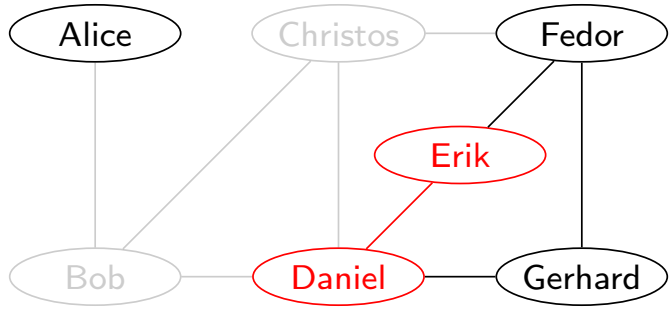

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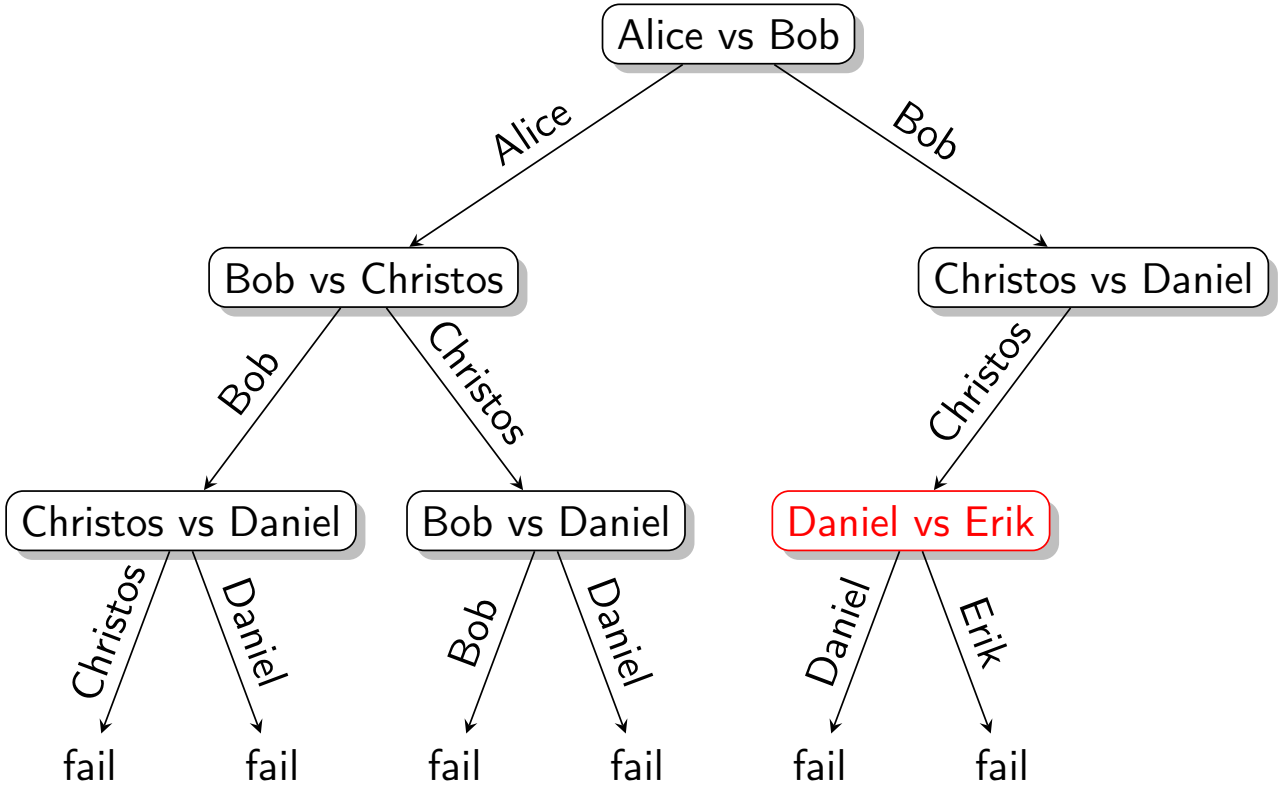

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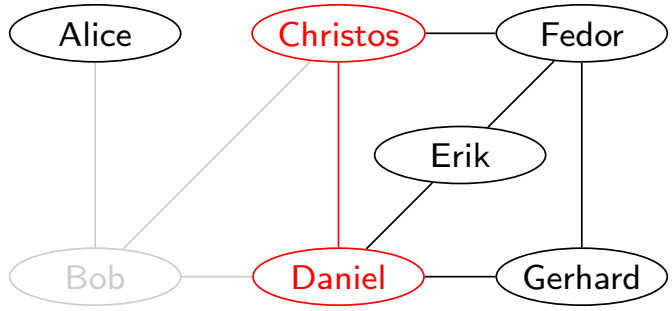

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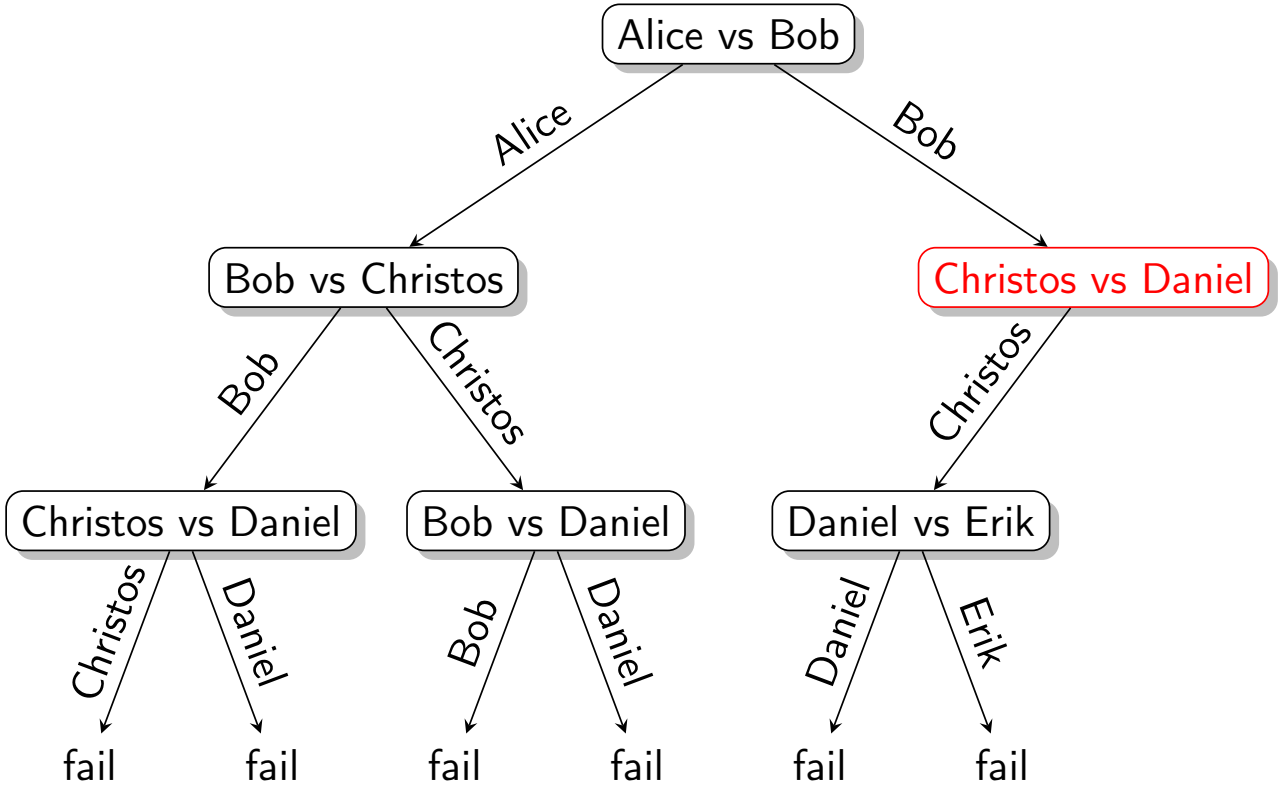

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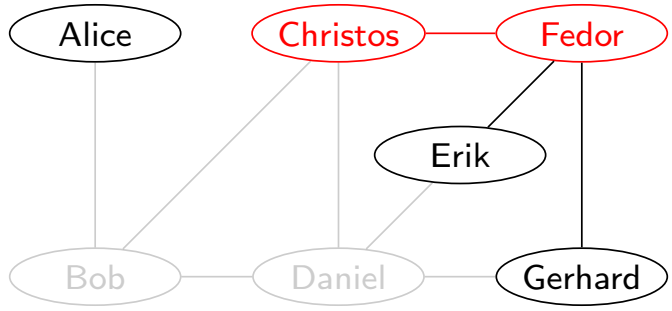

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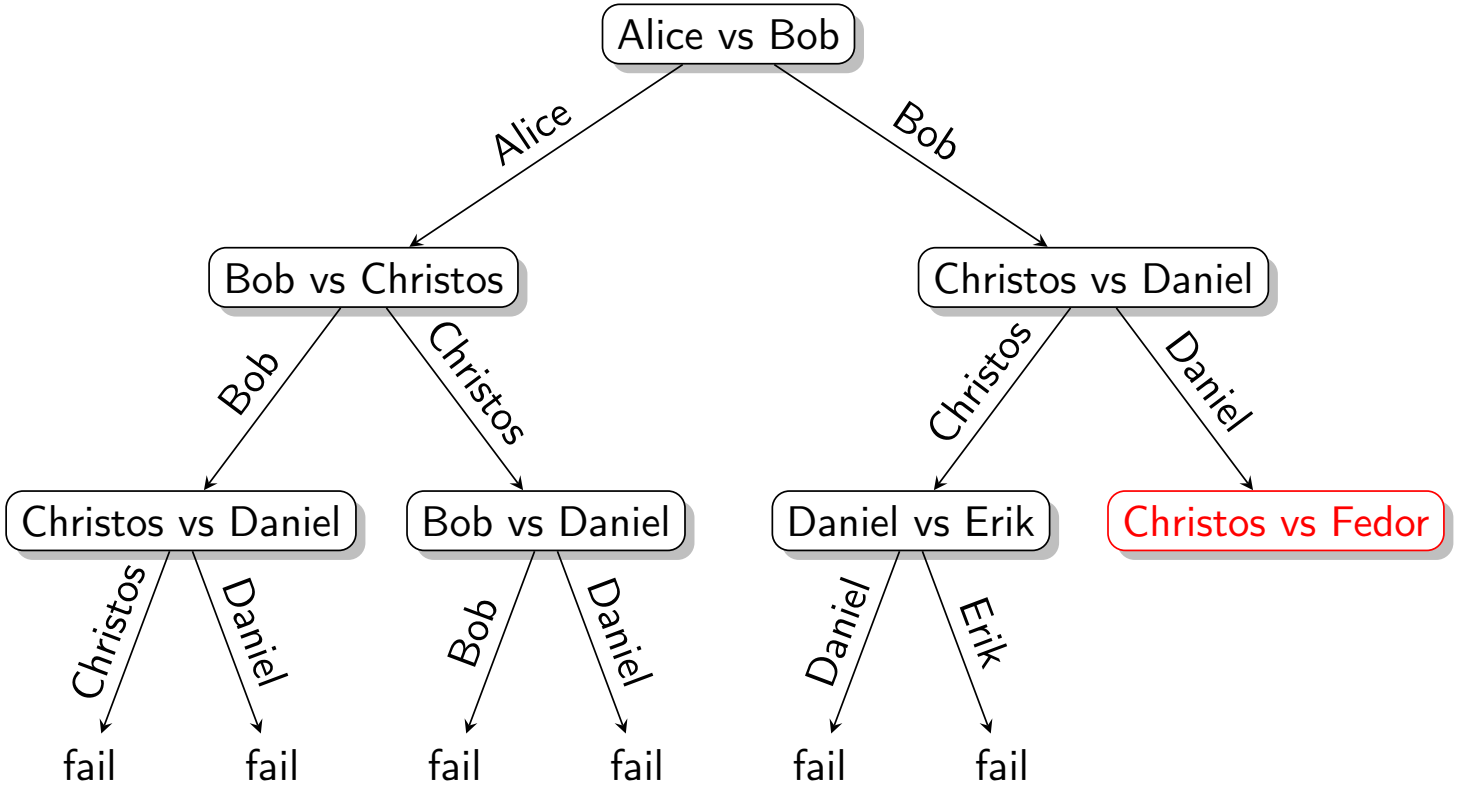

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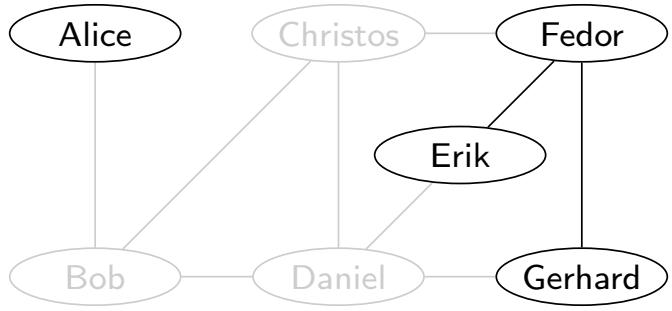

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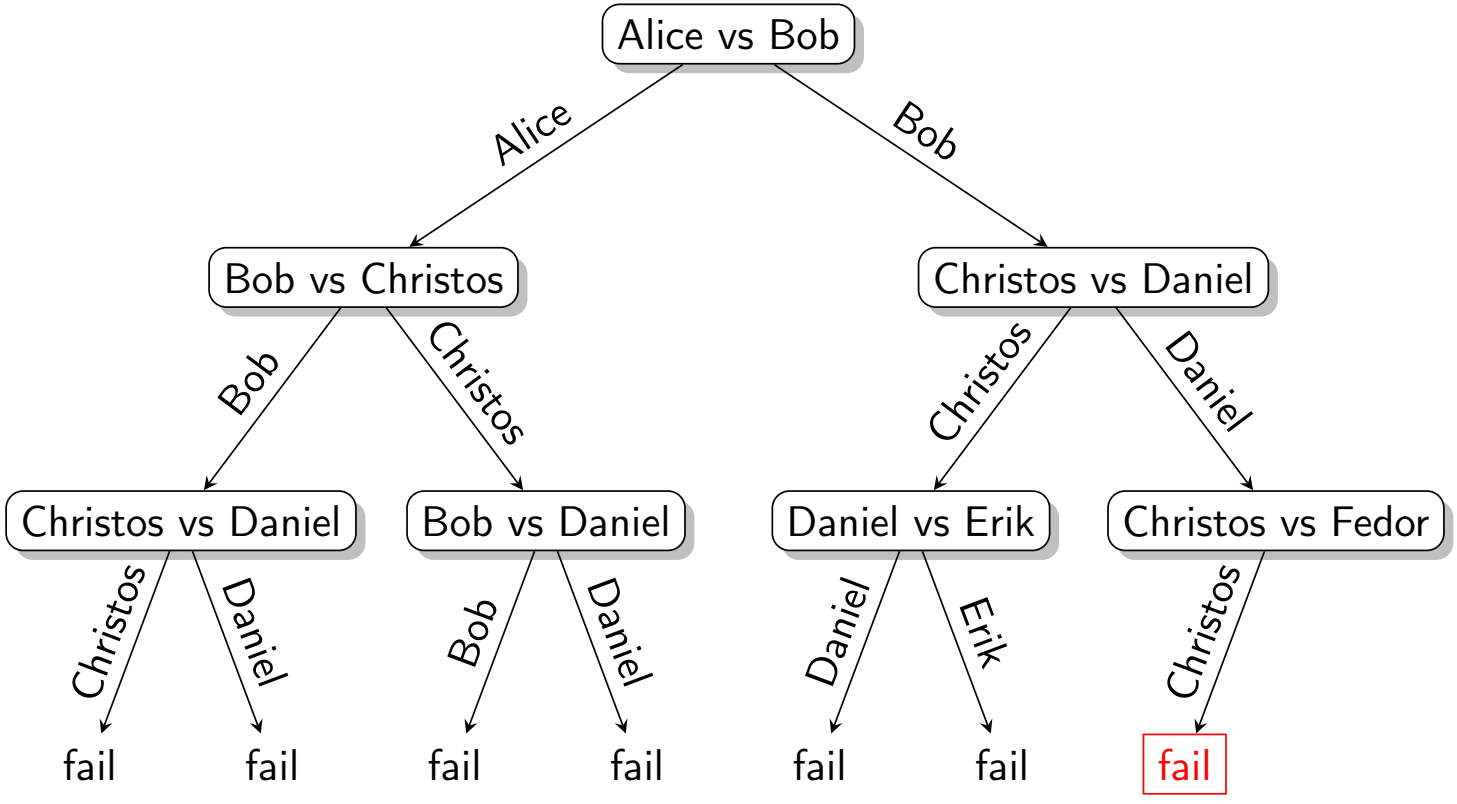

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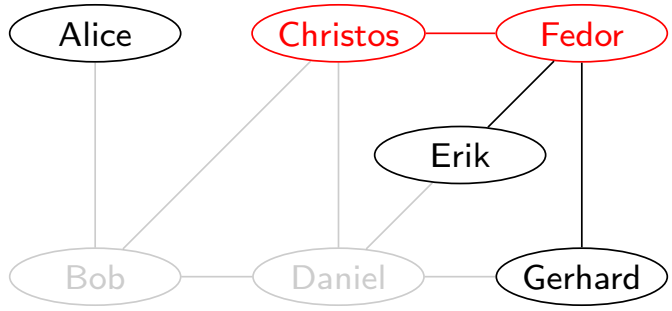

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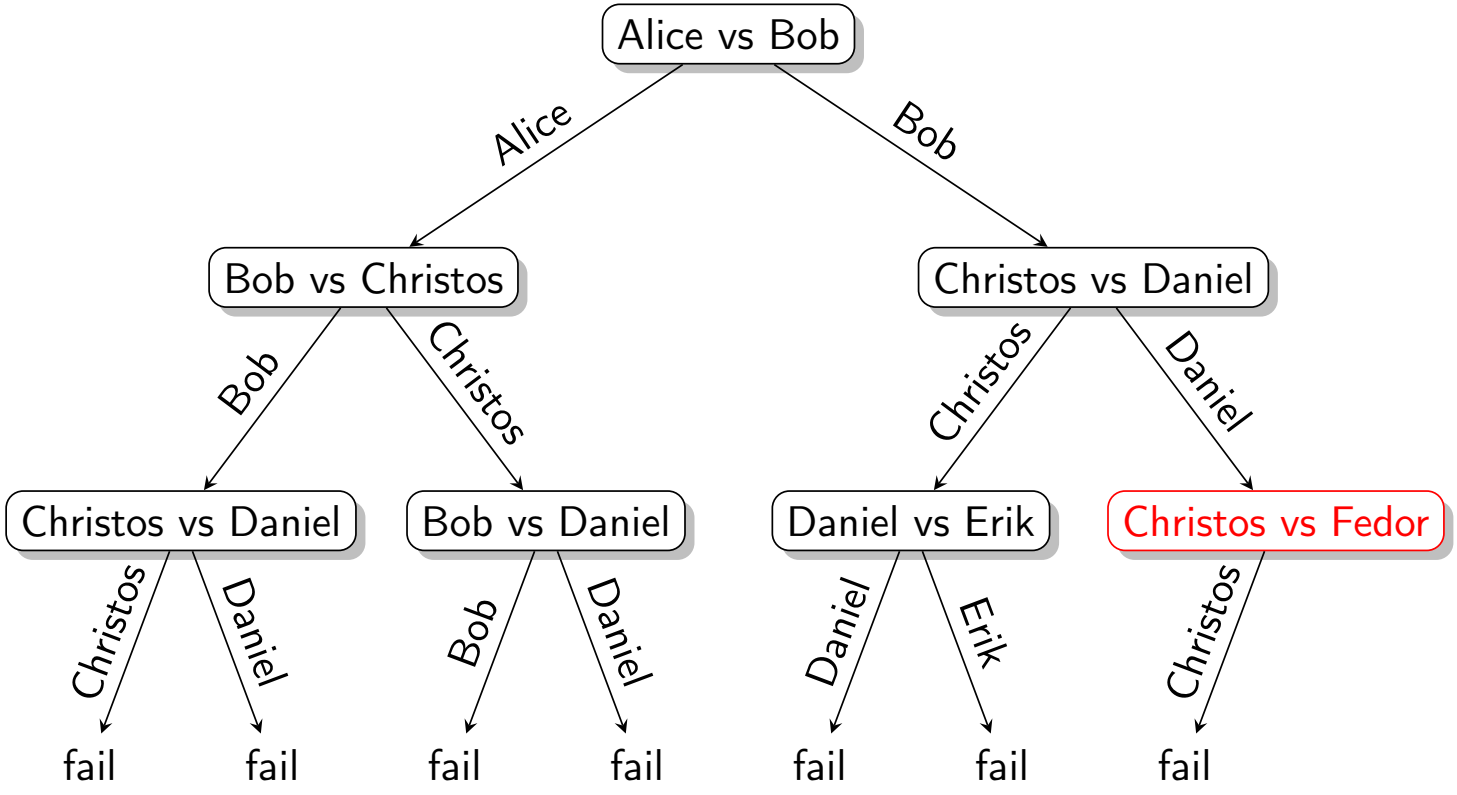

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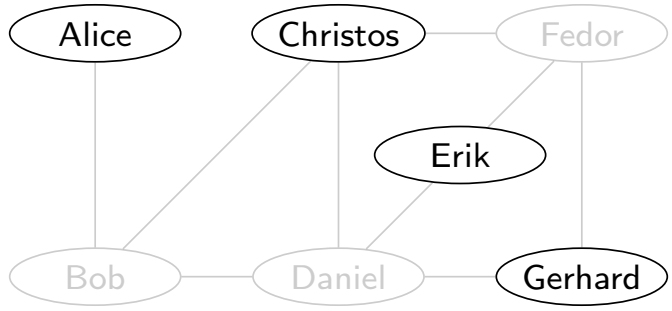

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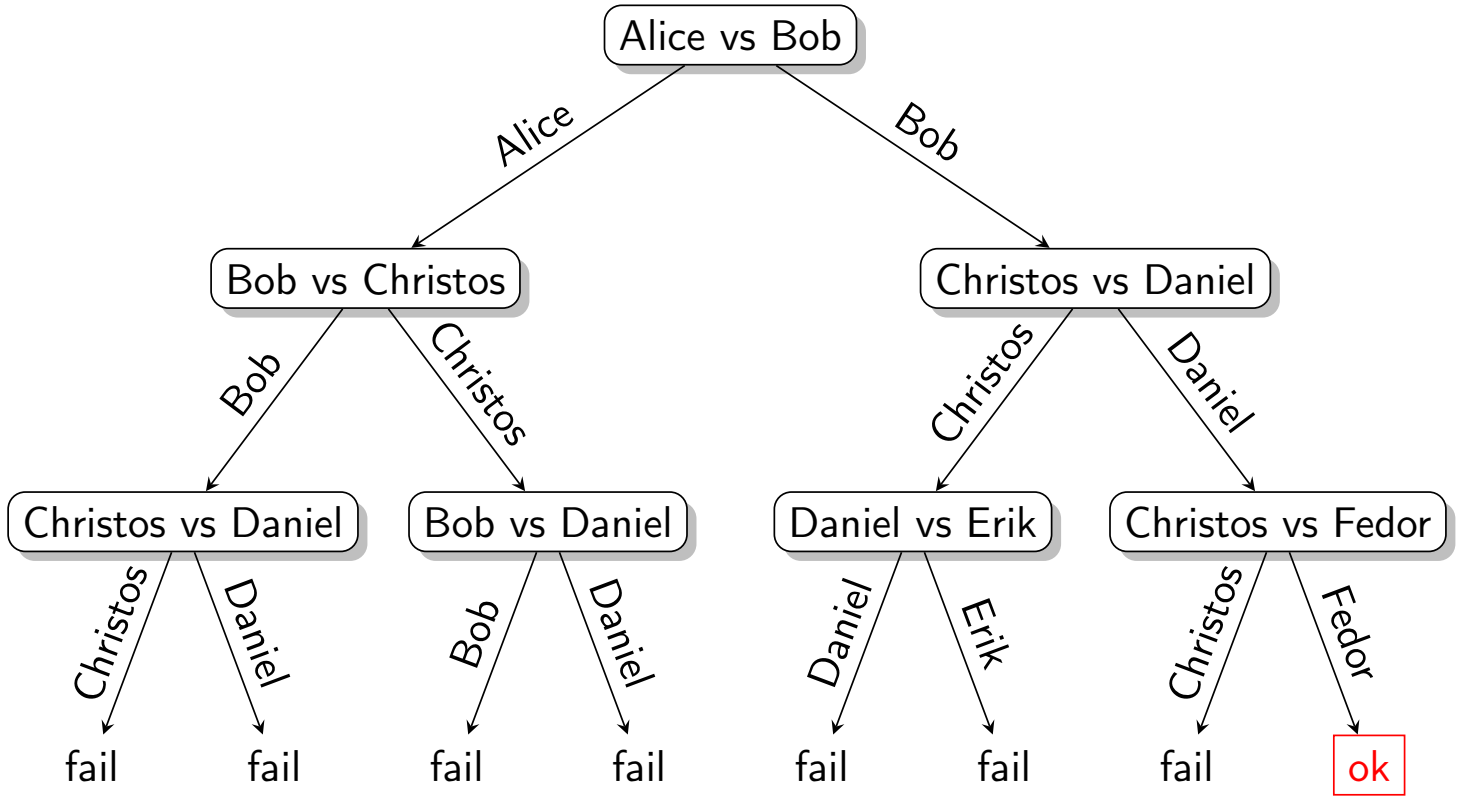

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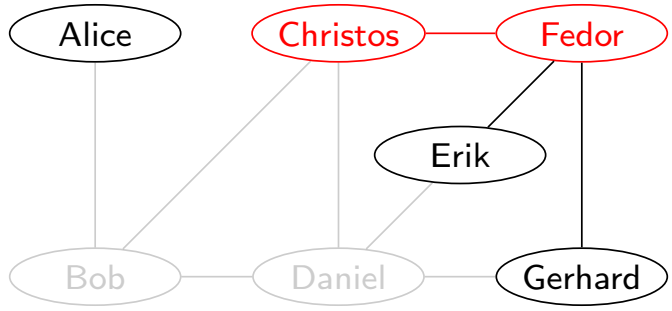

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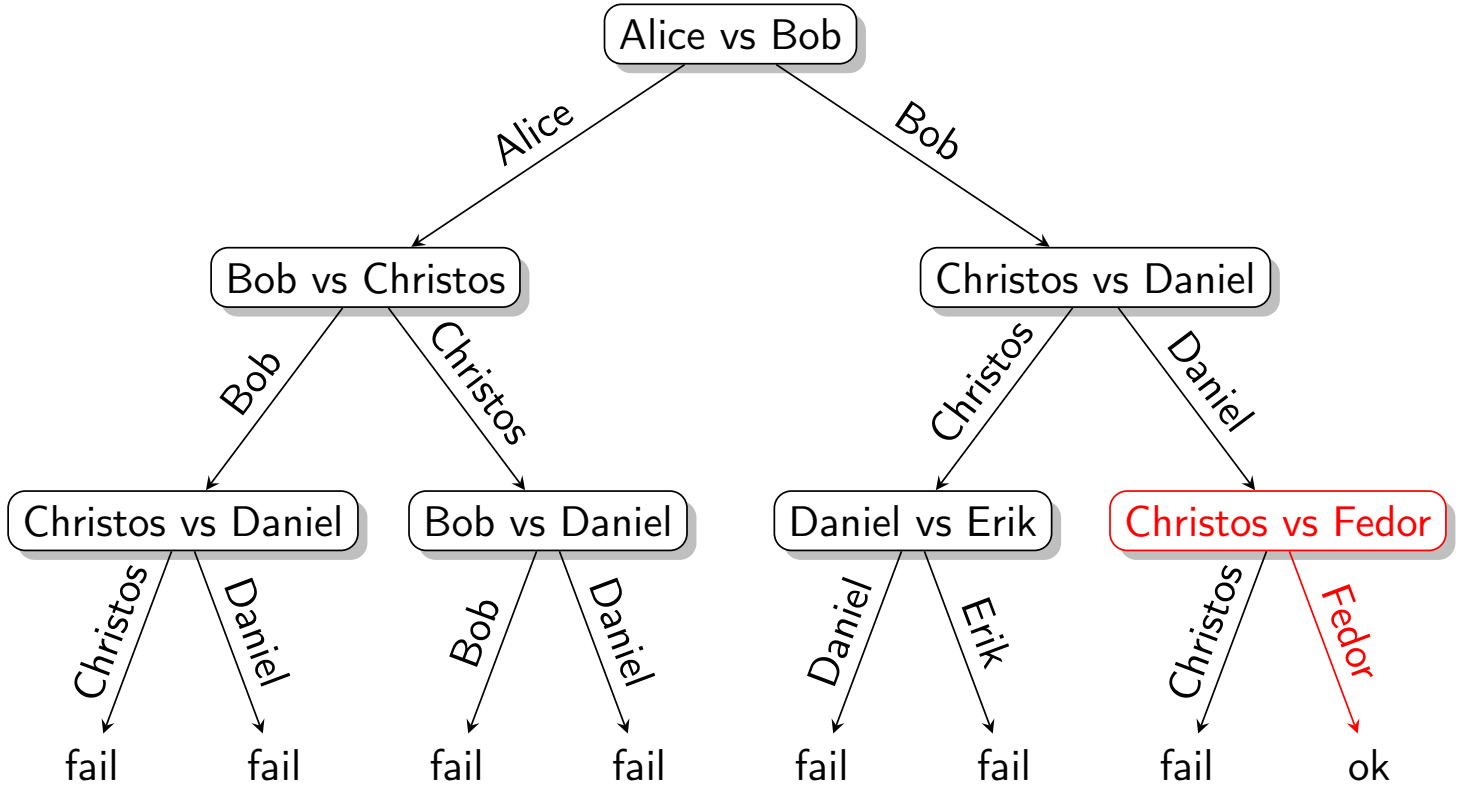
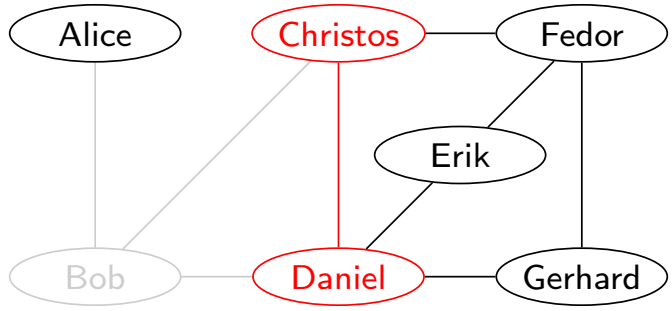
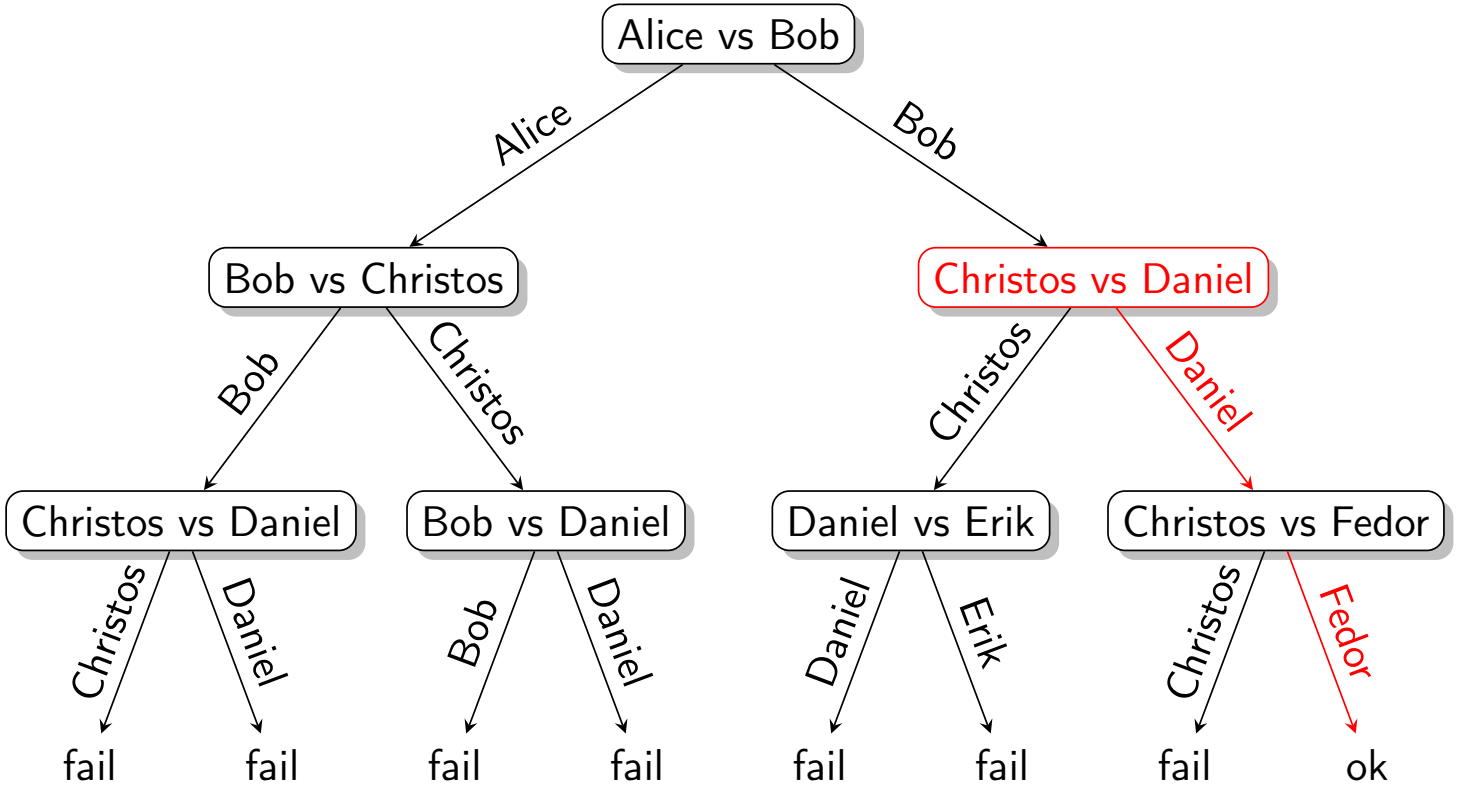


“Bar fight prevention” via Bounded Search Tree



$k = 2$ $C = \{\text{Fedor}, \text{Daniel}\}$



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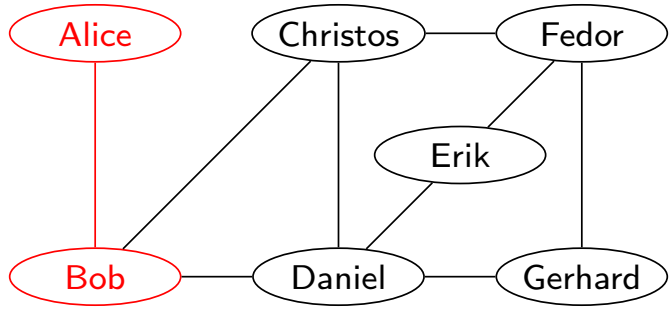
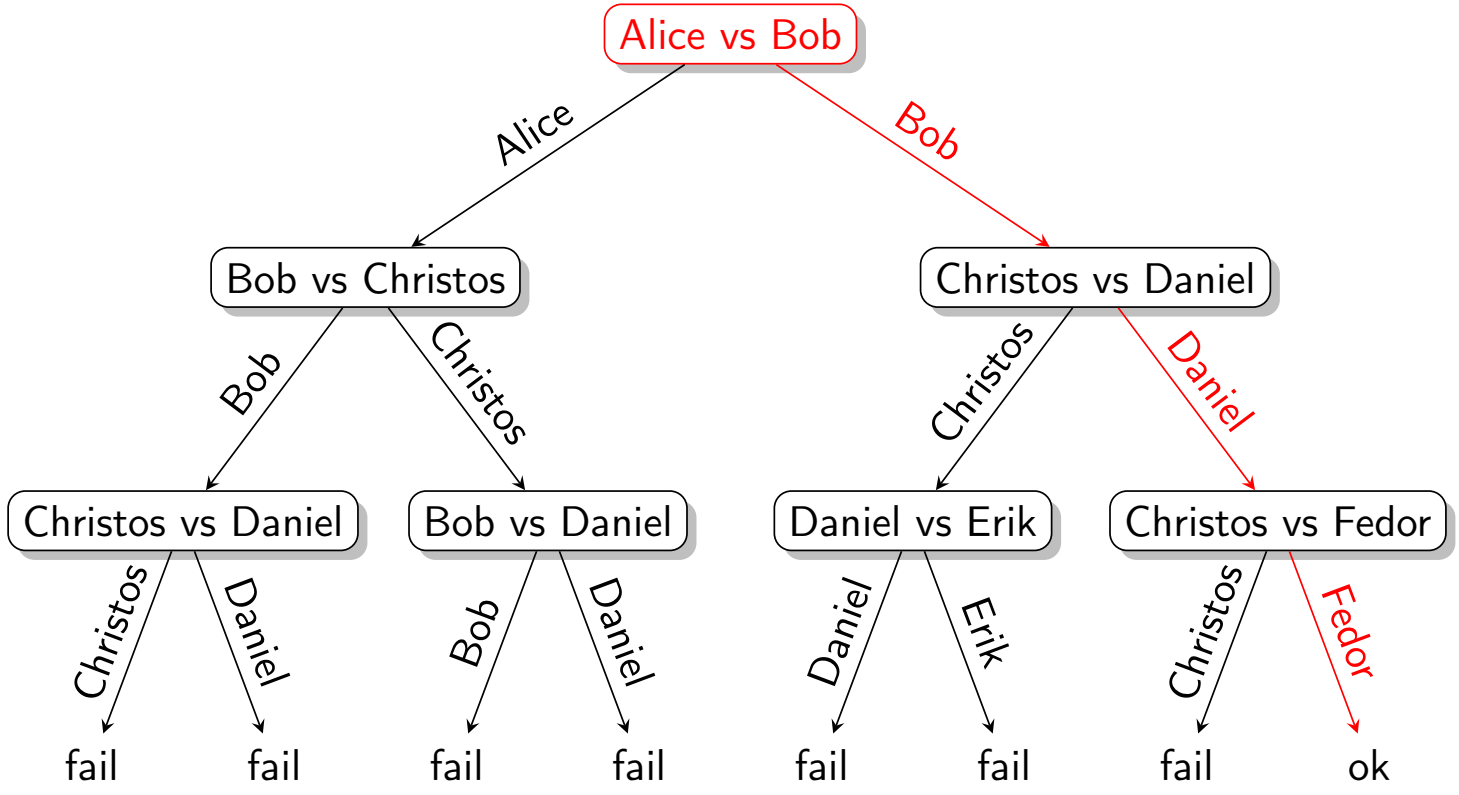


$k = 3$ $C = \{\text{Fedor}, \text{Daniel}, \text{Bob}\}$



AADS Lecture on EE/FPT, Part 4

FPT vs XP

FPT vs XP

An important feature of the Bar Fight Prevention problem is the existence of the *parameter* k. The problem of finding the minimum k that works is NP-complete, but for any fixed constant k we have just seen two linear-time algorithms!

We say the problem is *parameterized* by the *parameter* k. In this case k is the maximum solution size, but other problems may have different parameters (and may have more than one).

Definition: A parameterized problem is *Fixed Parameter Tractable (FPT)* if it is has an algorithm with running time $f(k) \cdot n^c$ for some function f and some constant $c \in \mathbb{R}$.

Definition: A parameterized problem is *Slice-wise Polynomial (XP)* if it is has an algorithm with running time $f(k) \cdot n^{g(k)}$ for some functions f, g.

Note: FPT $\subset$ XP, why?

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Note: FPT $\subset$ XP, why? Simply set $g(k) = c$.

Example: Vertex k-Coloring

Problem: For an integer k, given a graph G does G have a proper vertex coloring with k colors?

Lemma

Unless $P = NP$, this problem is not XP and therefore not FPT.

Proof sketch.

The problem is NP-hard even for $k = 5$, so unless $P = NP$ there can be no algorithm for general k with running time $f(k) \cdot n^{g(k)}$. □

Example: k-Clique

Problem: For an integer k, given a graph G does G have a clique of size k?

Lemma

k-clique is XP.

Proof.

A simple brute-force algorithm is to check every k-subset of the vertices. There are $\binom{n}{k} \leq n^k$ such subsets, and we can check in $\mathcal{O}(k^2)$ time whether a given subset forms a clique. Thus the running time of this algorithm is $\mathcal{O}(k^2 \cdot n^k)$ which proves the problem is in XP. □

It is unknown whether k-clique is FPT, but it is widely believed that $\mathcal{O}(n^k)$ is optimal which would prove it is not.

Example: Clique parameterized by Δ

Problem: For integer Δ, given a graph G with maximum degree Δ and an integer k, does G have a clique of size k?

Lemma

Clique is FPT when parameterized by the maximum degree Δ.

Proof.

If $k > \Delta + 1$ the answer is just no. Otherwise a naive algorithm is for each vertex to try all subsets of its neighbors. There are at most $n \cdot 2^{\Delta}$ such subsets and each can be checked in $\mathcal{O}(\Delta^2)$ time. The total time is thus $\mathcal{O}((2^{\Delta} \cdot \Delta^2) \cdot n)$, which proves the problem is FPT. □

In fact, we can easily improve this algorithm to run in $\mathcal{O}(\binom{\Delta}{k-1} \cdot k^2 \cdot n)$ time when $k \leq \Delta + 1$.

There are often many possible choices of parameter. Choosing the right one for a specific problem is an art.

Summary

Todays topics were "Exact exponential algorithms" and "Parameterized Complexity". We have covered

- The natural brute force algorithm for problems in NP.
- An exact $\mathcal{O}^*(2^n)$-time dynamic programming algorithm for TSP.
- An exact $\mathcal{O}^*(3^{n/3})$-time branching algorithm for MIS.
- A kernelization for the "Bar Fight Prevention" problem, a.k.a. k-vertex cover.
- A bounded search tree algorithm for k-vertex cover.
- Definitions of parameterized complexity, FPT and XP.
- Examples of problems in FPT, XP but not FPT, and not XP.
- Next time: van Emde Boas trees / predecessor search.

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Advanced algorithms and data structures

Lecture: van Emde Boas Trees

Jacob Holm (jaho@di.ku.dk)

January 3rd 2024

Today's Lecture

van Emde Boas Trees

- Predecessor search/ordered sets
- Naive
- Twolevel
- Recursive
- vEB: worst case $\mathcal{O}(\log\log|U|)$ time
- RS-vEB: expected $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(n\log\log|U|)$ space
- R^2S-vEB: expected $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(n)$ space
- Bonus: vEB is optimal for $w = \Theta(\log n)$
- Bonus: Integer sorting in expected $\mathcal{O}(n\log\log|U|)$ time.

Predecessor search/ordered sets

Problem:
Given a universe $U = [u]$ where $u = 2^w$,
maintain subset $S \subseteq U$, $|S| = n$ under:

member(x, S): Return $[x \in S]$.

insert(x, S): Add x to S (assumes $x \notin S$).

delete(x, S): Remove x from S (assumes $x \in S$).

empty(S): Return $[S = \emptyset]$.

min(S): Return $\min S$ (assumes $S \neq \emptyset$).

max(S): Return $\max S$ (assumes $S \neq \emptyset$).

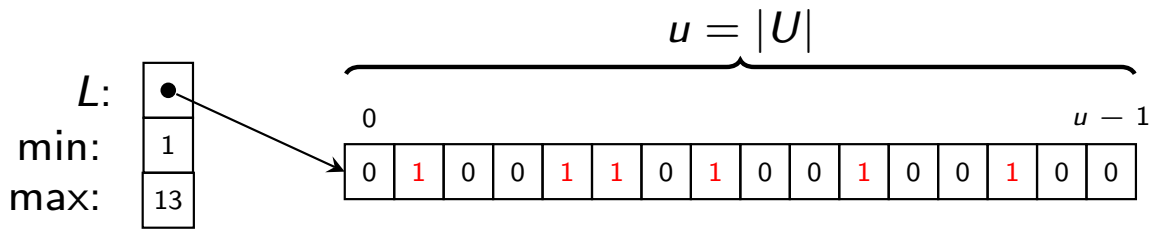
predecessor(x, S): Return $\max\{y \in S \mid y < x\}$
(assumes $\{y \in S \mid y < x\}$ is nonempty,
i.e. $S \neq \emptyset$ and $x > \min(S)$).

successor(x, S): Return $\min\{y \in S \mid y > x\}$
(assumes $\{y \in S \mid y > x\}$ is nonempty,
i.e. $S \neq \emptyset$ and $x < \max(S)$).

Naive

Idea: If we are willing to spend $\mathcal{O}(|U|)$ space...

Store S as a bit-array L of length $|U|$ such that $L[x] = [x \in S]$, and keep track of the min and max values explicitly.



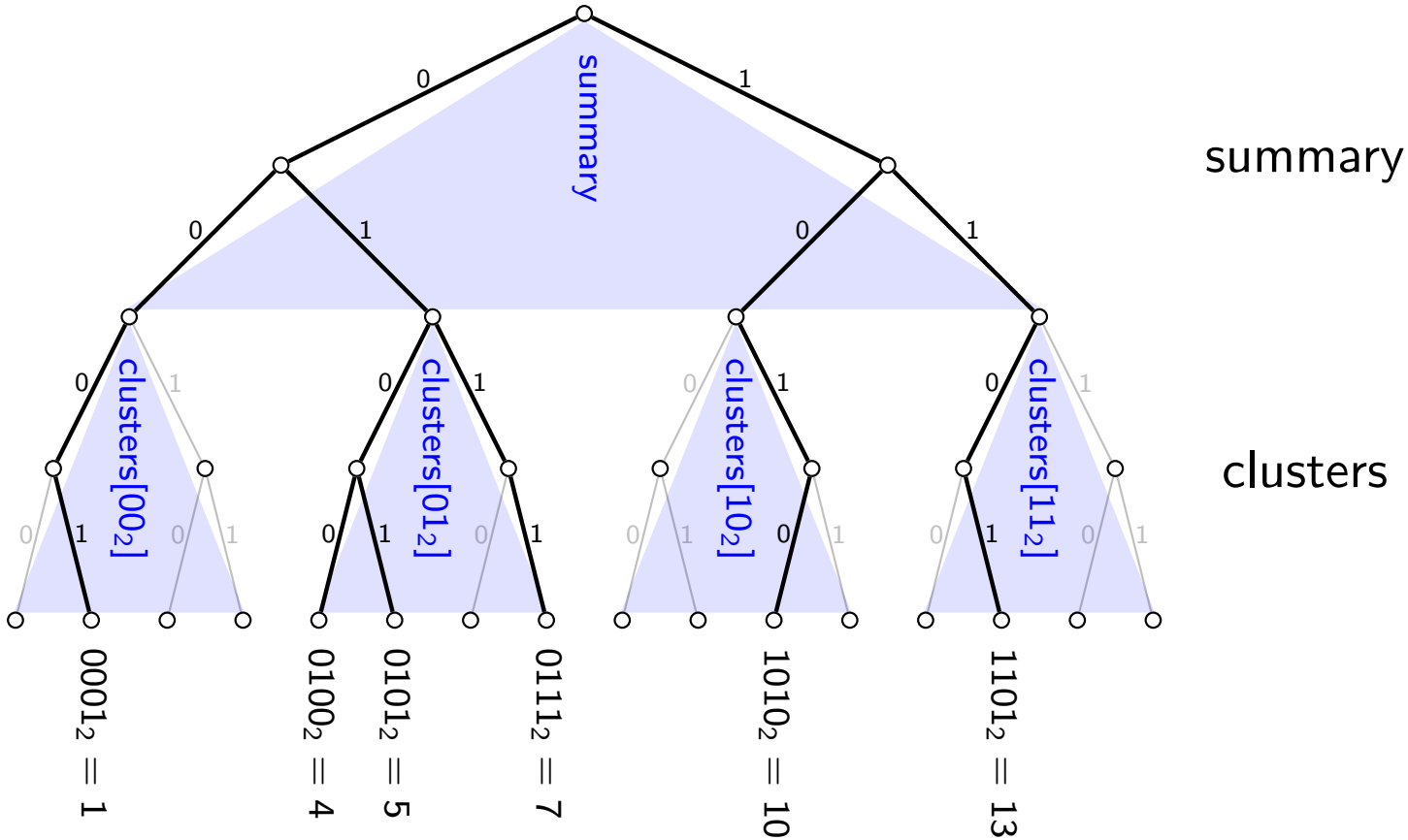
$\{1, 4, 5, 7, 10, 13\}$

How fast is:
empty(S), min(S), and max(S)? worst case $\mathcal{O}(1)$.
member(x, S)? worst case $\mathcal{O}(1)$.
predecessor(x, S) and successor(x, S)? worst case $\Theta(|U|)$.
delete(x, S)? worst case $\mathcal{O}(|U|)$.
insert(x, S)? worst case $\mathcal{O}(1)$.

Bit-Trie

Idea: Think of each key as a w-bit string describing a path in a binary *trie* (= a special kind of tree). The naive structure ignores all intermediate branches and jump directly to the leaves. An alternative naive choice would be to store the actual trie, which we could maintain in $\mathcal{O}(w) = \mathcal{O}(\log|U|)$ time.

What if we instead split each key into a high and a low part?



Twolevel

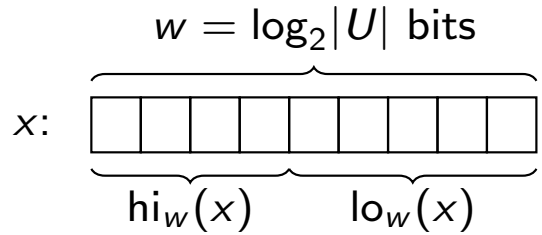
Idea: Split each key into *high* and *low* parts. Use naive for each.

Recall that $U = [2^w]$, and define:

$$\mathsf{hi}_w(x) := \left\lfloor \frac{x}{2^{\lceil w/2 \rceil}} \right\rfloor$$

$$\mathsf{lo}_w(x) := x \bmod 2^{\lceil w/2 \rceil}$$

$$\mathsf{index}_w(h, \ell) := h \cdot 2^{\lceil w/2 \rceil} + \ell$$



note that $x = \mathsf{index}_w(\mathsf{hi}_w(x), \mathsf{lo}_w(x))$.

Now let the structure directly store the values:

$$\mathsf{min} := \min(S) \text{ if } S \neq \emptyset, \text{ else } 1$$

$$\mathsf{max} := \max(S) \text{ if } S \neq \emptyset, \text{ else } 0$$

and use the naive structure to store

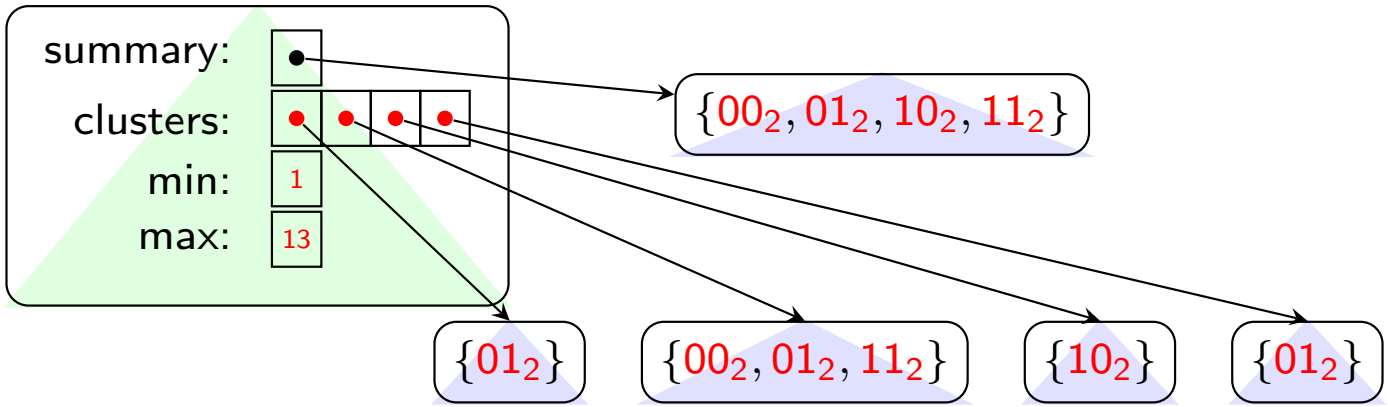
$$\mathsf{summary} := \{\mathsf{hi}_w(x) \mid x \in S\}$$

$$\mathsf{clusters}[h] := \{\ell \in [2^{\lceil w/2 \rceil}] \mid \mathsf{index}_w(h, \ell) \in S\} \qquad \forall h \in [2^{\lfloor w/2 \rfloor}]$$

note that $S = \bigcup_{h \in [2^{\lfloor w/2 \rfloor}]} \{\mathsf{index}_w(h, \ell) \mid \ell \in \mathsf{clusters}[h]\}$.

Twolevel

We can draw the structure for the set $S = \{1, 4, 5, 7, 10, 13\} = \{0001_2, 0100_2, 0101_2, 0111_2, 1010_2, 1101_2\} \subseteq [2^4]$ as:



How fast (worst case) is:

empty(S), min(S), max(S)?
$\mathcal{O}(1)$
member(x, S)?
$\mathcal{O}(1) + 1 \times \text{naive} = \mathcal{O}(1)$
predecessor(x, S), successor(x, S)?
$\mathcal{O}(1) + 1 \times \text{naive}$
$= \Theta(2^{\lceil w/2 \rceil}) = \Theta(\sqrt{|U|})$
delete(x, S)?
$\mathcal{O}(1) + 2 \times \text{naive} = \Theta(\sqrt{|U|})$
insert(x, S)?
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function EMPTY(S)
  return S.min $>$ S.max
function MIN(S)
  ▷ Assumes $S \neq \emptyset$
  return S.min
function MAX(S)
  ▷ Assumes $S \neq \emptyset$
  return S.max
function MEMBER$_w$(x, S)
  return MEMBER$_{\lceil w/2 \rceil}$(lo$_w(x)$, S.clusters[hi$_w(x)$])

How can we improve the time further?

Twolevel

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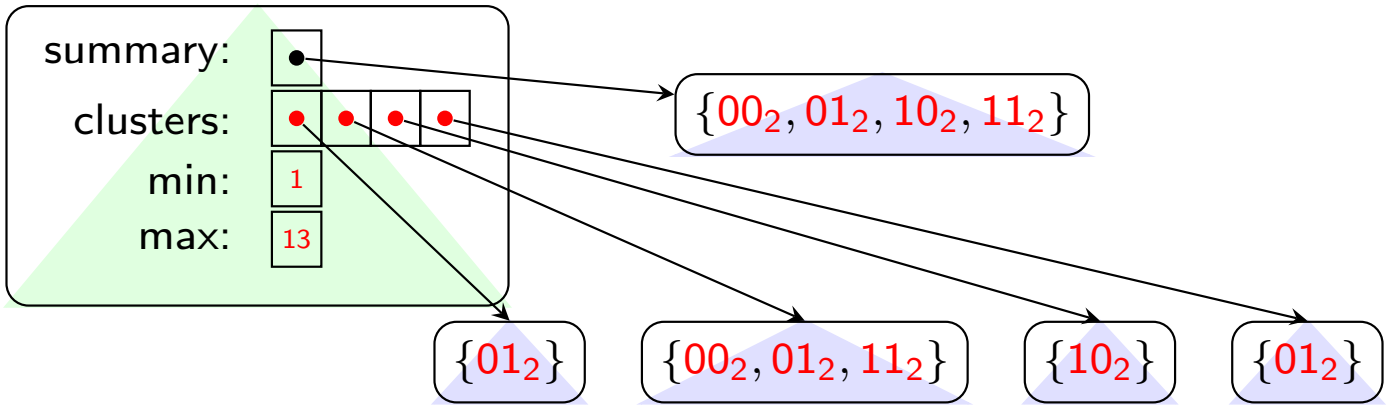

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function $\textsc{predecessor}_w(x, S)$
  ▷ Assumes $S \neq \emptyset$ and $\min(S) < x$
  if $x > S.\max$ **then**
    return $S.\max$
  $p \leftarrow \text{hi}_w(x)$, $s \leftarrow \text{lo}_w(x)$, $C \leftarrow S.\text{clusters}[p]$
  if not $\textsc{empty}(C)$ and $C.\min < s$ **then**
    return $\text{index}_w(p, \textsc{predecessor}_{\lceil w/2 \rceil}(s, C))$
  $p \leftarrow \textsc{predecessor}_{\lfloor w/2 \rfloor}(p, S.\text{summary})$
  return $\text{index}_w(p, S.\text{clusters}[p].\max)$

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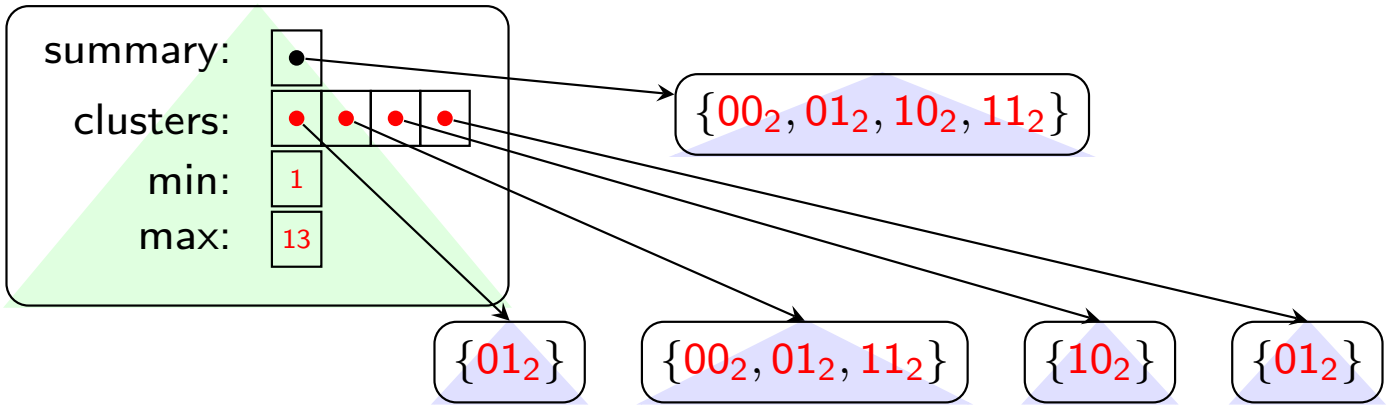

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function $\text{DELETE}_w(x, S)$
  ▷ Assumes $x \in S$
  $p \leftarrow \text{hi}_w(x)$, $s \leftarrow \text{lo}_w(x)$, $C \leftarrow S.\text{clusters}[p]$
  $\text{DELETE}_{\lceil w/2 \rceil}(s, C)$
  if $\text{EMPTY}(C)$ **then**
    $\text{DELETE}_{\lfloor w/2 \rfloor}(p, S.\text{summary})$
  if $S.\min = S.\max$ **then**
    $S.\min \leftarrow 1$, $S.\max \leftarrow 0$
  else if $x = S.\min$ **then**
    $p \leftarrow S.\text{summary.min}$, $S.\min \leftarrow \text{index}_w(p, S.\text{clusters}[p].\min)$
  else if $x = S.\max$ **then**
    $p \leftarrow S.\text{summary.max}$, $S.\max \leftarrow \text{index}_w(p, S.\text{clusters}[p].\max)$

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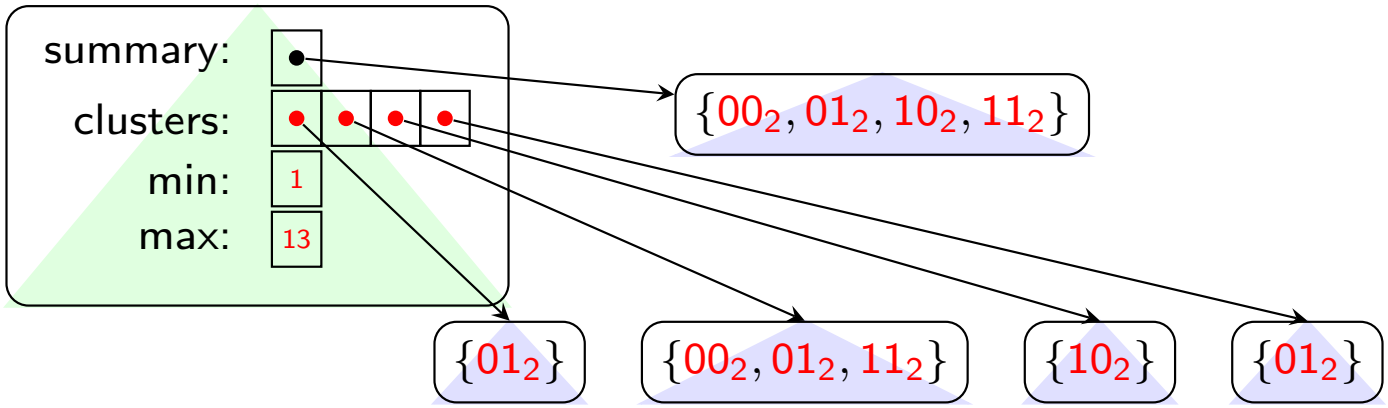

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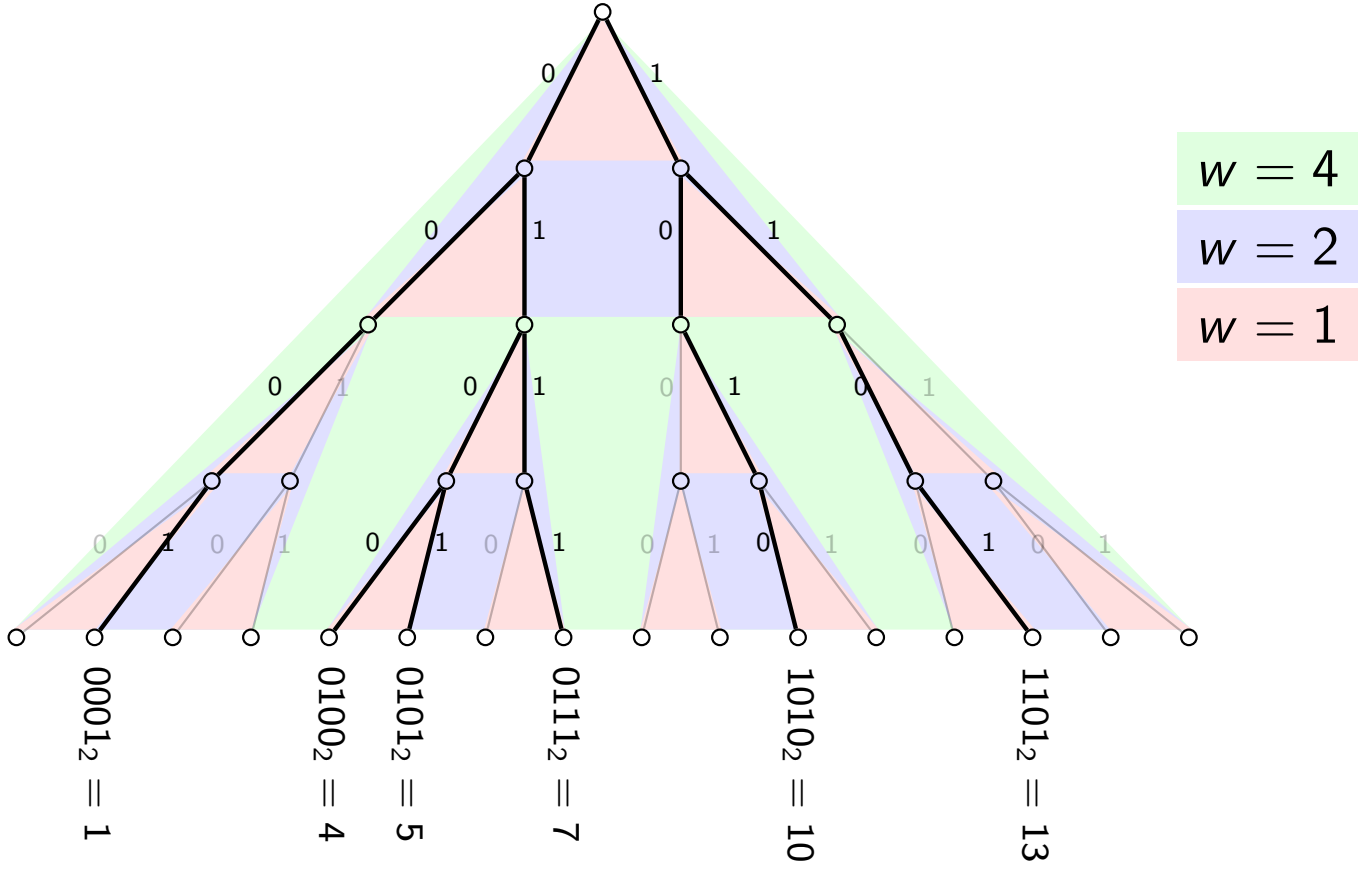
delete(x, S)?
$\mathcal{O}(1) + 2 \times \text{naive} = \Theta(\sqrt{|U|})$
insert(x, S)?
$\mathcal{O}(1) + 2 \times \text{naive} = \mathcal{O}(1)$

function INSERT$_w(x, S)$
▷ Assumes $x \notin S$
if EMPTY(S) **then**
S.min $\leftarrow x$, S.max $\leftarrow x$
if $x < S$.min **then** S.min $\leftarrow x$
if $x > S$.max **then** S.max $\leftarrow x$
$p \leftarrow \text{hi}_w(x)$, $s \leftarrow \text{lo}_w(x)$
if EMPTY(S.clusters[p]) **then**
INSERT$_{\lfloor w/2 \rfloor}(p, S.\text{summary})$
INSERT$_{\lceil w/2 \rceil}(s, S.\text{clusters}[p])$

How can we improve the time further?

Recursive

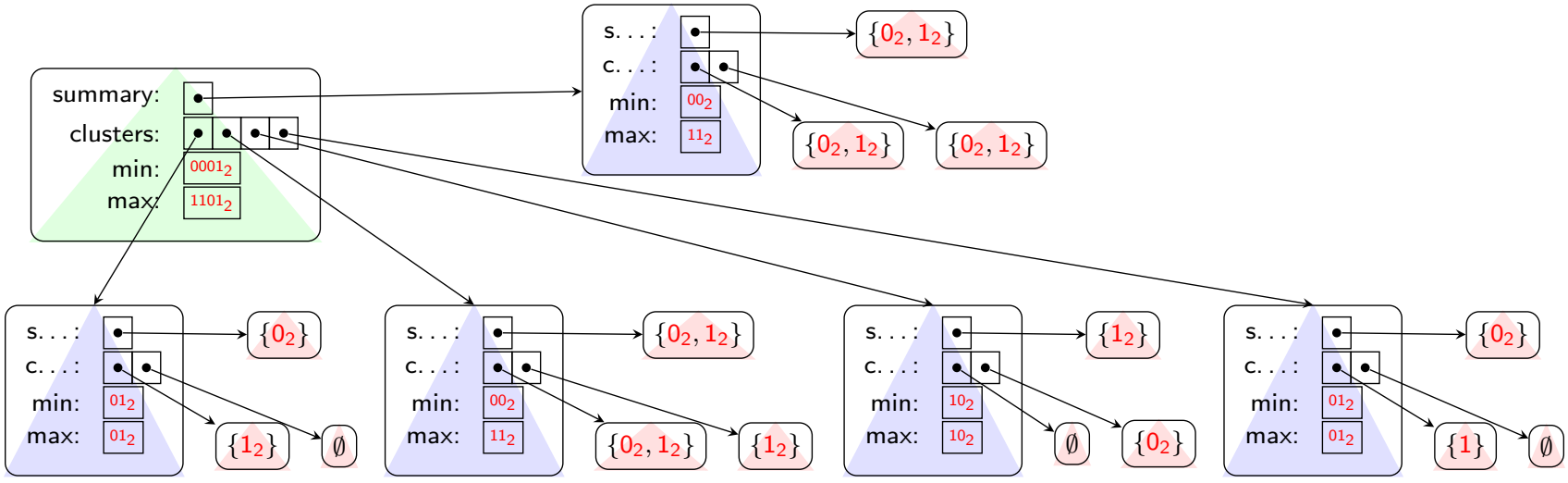
Idea: Instead of using the naive structure for summary and clusters[h], recursively use the same type of structure (stop recursion when $w = 1$).



Recursive

Idea: Instead of using the naive structure for summary and clusters[h], recursively use the same type of structure (stop recursion when $w = 1$).

We can draw the recursive structure for the set $S = \{1, 4, 5, 7, 10, 13\} = \{0001_2, 0100_2, 0101_2, 0111_2, 1010_2, 1101_2\} \subseteq [2^4]$ as:



This structure is (essentially) what the book calls "proto-vEB". Note that the naive structure we started with is completely gone.

Recursive

Theorem

The recursion depth of this structure, when used on the universe $U = [2^w]$ *is* $\lceil \log_2 w \rceil = \mathcal{O}(\log \log |U|)$.

Proof.

Let $d(w)$ be the recursion depth when working with a universe of size 2^w. We will prove by induction that $d(w) = \lceil \log_2 w \rceil$.
The base case is $w = 1$ where there is no recursion, so $d(1) = 0 = \lceil \log_2(1) \rceil$.
For the induction case, suppose $w > 1$ and that $d(w') = \lceil \log_2(w') \rceil$ for all $w' \in [w]_+$. Now let $w' = \lceil w/2 \rceil \in [w]_+$, then the largest universe size used in the recursion is $2^{w'}$, and (by induction) $d(w) = 1 + d(w') = 1 + \lceil \log_2(w') \rceil = \lceil \log_2(2w') \rceil$. What remains is to show $\lceil \log_2(2w') \rceil = \lceil \log_2(w) \rceil$.

If w is even, $2w' = w$ and we are done.
Otherwise $w > 1$ is odd so $w \geq 3$ and the smallest integer $k = \lceil \log_2(w) \rceil$ such that $2^k \geq w$ satisfies $k \geq 1$. Thus 2^k is even and so must satisfy $2^k \geq w + 1 = 2w'$ and therefore also $k = \lceil \log_2(2w') \rceil = \lceil \log_2(w) \rceil$. □

Recursive

Using the recursive structure, how fast is:
empty(S), min(S), and max(S)? worst case $\mathcal{O}(1)$.
member(x, S)? $\mathcal{O}(1) + 1\times$recursion $= \mathcal{O}(d(w)) = \mathcal{O}(\log\log|U|)$.
predecessor(x, S) and successor(x, S)? $O(1) + 1\times$recursion
$= \mathcal{O}(d(w)) = \mathcal{O}(\log\log|U|)$.
insert(x, S) and delete(x, S)? $\mathcal{O}(1) + 2\times$recursion
$= \Theta(2^{d(w)}) = \Theta(w) = \Theta(\log|U|)$.

How can we improve the update time? Somehow make insert and delete recurse in only one substructure.

vEB: worst case $\mathcal{O}(\log\log|U|)$ time

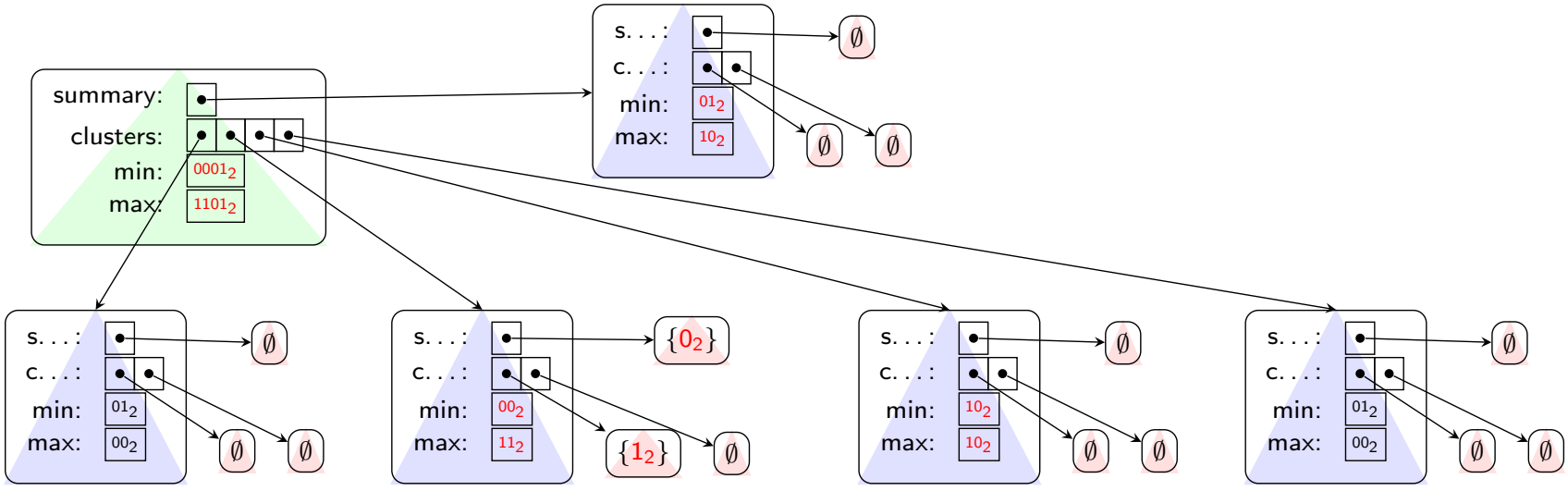
Idea: Exclude $\min(S)$ and/or $\max(S)$ from the set of keys stored in summary and clusters. (CLRS excludes only $\min(S)$, I exclude both).

Specifically, redefine summary and clusters to recursively store the sets:

$$\text{summary} := \{\text{hi}_w(x) \mid x \in S \setminus \{\min, \max\}\}$$

$$\text{clusters}[h] := \{\ell \in [2^{\lceil w/2 \rceil}] \mid \text{index}_w(h, \ell) \in S \setminus \{\min, \max\}\} \quad \forall h \in [2^{\lfloor w/2 \rfloor}]$$

We can draw the van Emde Boas tree for the set $S = \{1, 4, 5, 7, 10, 13\} = \{0001_2, 0100_2, 0101_2, 0111_2, 1010_2, 1101_2\} \subseteq [2^4]$ as:



vEB: worst case $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(|U|)$ space

1: **function** $\textsc{predecessor}_w(x, S)$ ▷ Assumes $S \neq \emptyset$ and $x > S.\min$
2: **if** $x > S.\max$ **then**
3: **return** $S.\max$
4: **if** $w = 1$ **then**
5: **return** $S.\min$
6: $p \leftarrow \text{hi}_w(x)$, $s \leftarrow \text{lo}_w(x)$, $C \leftarrow S.\text{clusters}[p]$
7: **if** not $\textsc{empty}(C)$ and $C.\min < s$ **then**
8: **return** $\text{index}_w(p, \textsc{predecessor}_{\lceil w/2 \rceil}(s, C))$
9: **if** $\textsc{empty}(S.\text{summary})$ or $p \leq S.\text{summary}.\min$ **then**
10: **return** $S.\min$
11: $p \leftarrow \textsc{predecessor}_{\lfloor w/2 \rfloor}(p, S.\text{summary})$
12: **return** $\text{index}_w(p, S.\text{clusters}[p].\max)$

Theorem

$\textsc{predecessor}_w(x, S)$ *takes worst case* $\mathcal{O}(d(w)) = \mathcal{O}(\log\log|U|)$ *time.*

Proof.

It makes at most one recursive call. □

vEB: worst case $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(|U|)$ space

```
1:  function INSERT_w(x, S)                          ▷ Assumes x ∉ S
2:      if EMPTY(S) then
3:          S.min ← x, S.max ← x, return
4:      if S.min = S.max then
5:          if x < S.min then S.min ← x
6:          if x > S.max then S.max ← x
7:          return
8:      if x < S.min then S.min ↔ x
9:      if x > S.max then S.max ↔ x
10:     p ← hi_w(x), s ← lo_w(x)
11:     if EMPTY(S.clusters[p]) then
12:         INSERT_⌊w/2⌋(p, S.summary)
13:     INSERT_⌈w/2⌉(s, S.clusters[p])
```

Theorem

$\text{INSERT}_w(x, S)$ *takes worst case* $\mathcal{O}(d(w)) = \mathcal{O}(\log\log|U|)$ *time.*

Proof.

It makes at most one recursive call on a non-empty substructure, and inserting in an empty substructure takes constant time. □

RS-vEB: expected $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(n\log\log|U|)$ space

Idea: Use a hash table instead of an array to store clusters[·], and don't store empty substructures.

Why does this change updates from worst case $\mathcal{O}(\log\log|U|)$ to expected $\mathcal{O}(\log\log|U|)$ time? Because updates to a hash table take expected $\mathcal{O}(1)$ time rather than worst case.

Why does this only use $\mathcal{O}(n \cdot d(w)) = \mathcal{O}(n\log\log|U|)$ space? Because the empty structure uses $\mathcal{O}(1)$ space and INSERT_w only creates or updates $\mathcal{O}(d(w))$ substructures in the worst case. Each of these costs at most an additional $\mathcal{O}(1)$ space.

R^2S-vEB: expected $\mathcal{O}(\log\log|U|)$ time, $\mathcal{O}(n)$ space

Idea: Partition S into "chunks" of size $\Theta(\min\{n, \log\log|U|\})$, and store only one element representing each chunk in the RS-vEB structure. I.e. RS-vEB stores only $\mathcal{O}(\max\{1, n/\log\log|U|\})$ elements, using $\mathcal{O}(n)$ space.

We can store each chunk as a sorted linked list, and keep a hash table mapping each representative to its chunk. This also takes $\mathcal{O}(n)$ space.

PREDECESSOR and SUCCESSOR uses the RS-vEB structure to find the nearest two representatives in $\mathcal{O}(\log\log|U|)$ time, and can then spend linear time in the size of the two chunks to find the result.

INSERT may have to split a chunk that becomes too large and insert a new representative in the RS-vEB structure. Splitting the chunk can take linear time in the size of the chunk, and together with inserting the new representative into the RS-vEB structure, this still only takes expected $\mathcal{O}(\log\log|U|)$ time.

Similarly, DELETE may have to join two chunks and delete a representative, but again this only takes expected $\mathcal{O}(\log\log|U|)$ time.

Bonus: vEB is optimal for $w = \Theta(\log n)$

In fact, $\mathcal{O}(\log w)$ is the optimal query time when $w \in \Theta(\log n)$, or equivalently when $n \in 2^{\Theta(w)}$. This was shown in:

Time-space trade-offs for predecessor search,
Mihai Pătraşcu and Mikkel Thorup,
STOC'06: Proceedings of the thirty-eighth annual ACM symposium on Theory of Computing, pages 232–240.
`https://doi.org/10.1145/1132516.1132551`

Bonus: Integer sorting in expected $\mathcal{O}(n \log\log|U|)$ time.

Idea: Insert all elements in R^{2}S-vEB structure, use $\textsc{min}(S)$ then repeatedly $\textsc{successor}(x, S)$.

This takes expected $\mathcal{O}(n \log\log|U|) = \mathcal{O}(n \log w)$ time.

Compare this to other famous sorting methods:

QuickSort Takes expected $\mathcal{O}(n \log n)$ time.

CountingSort Takes $\mathcal{O}(n + |U|)$ time.

RadixSort Takes $\mathcal{O}((n + r) \log_r |U|) = \mathcal{O}((n + r)\frac{w}{\log r})$ time for radix r, e.g. $\mathcal{O}(\frac{n}{\log n} \log|U|) = \mathcal{O}(\frac{nw}{\log n})$ for $r = n$.

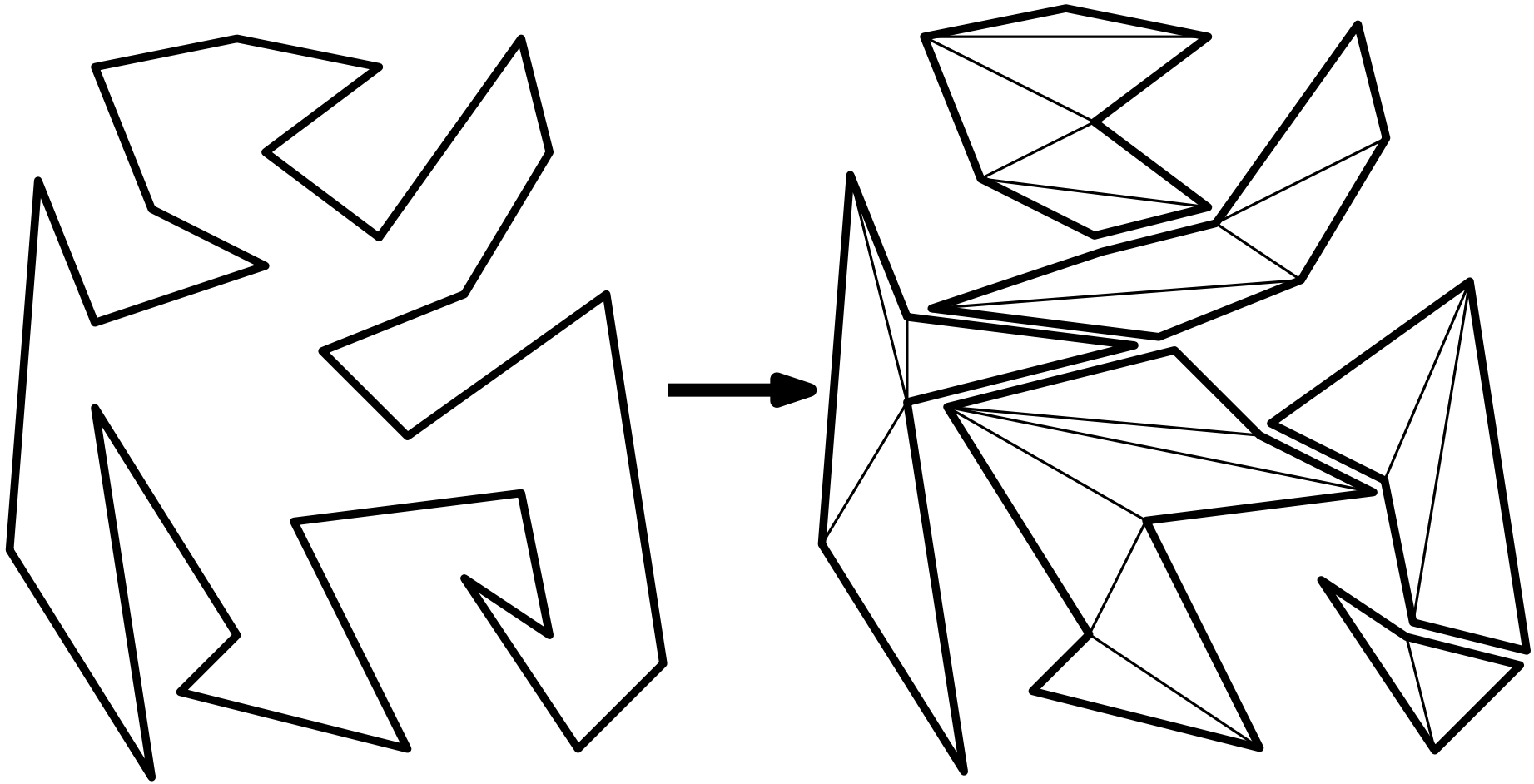
We observe that for $\log n \in \Omega(\log w) \cap \mathcal{O}(\frac{w}{\log w})$, or equivalently $n \in w^{\Omega(1)} \cap 2^{\mathcal{O}(\frac{w}{\log w})}$, sorting using vEB should be better than these alternatives.

Summary

Todays topic was van Emde Boas Trees, which is a data structure for sets of bounded integers. We have covered

- A naive implementation of ordered sets.
- A slightly less naive two-level implementation.
- A recursive implementation with good query time (proto-vEB).
- The actual van Emde Boas Tree (vEB).
- An extension to vEB using hashing to save space (RS-vEB).
- A further space-saving extension using "indirection" (R^2S-vEB).
- Some notes about the optimality of vEB and on integer sorting.
- Next time: Polygonal Triangulation

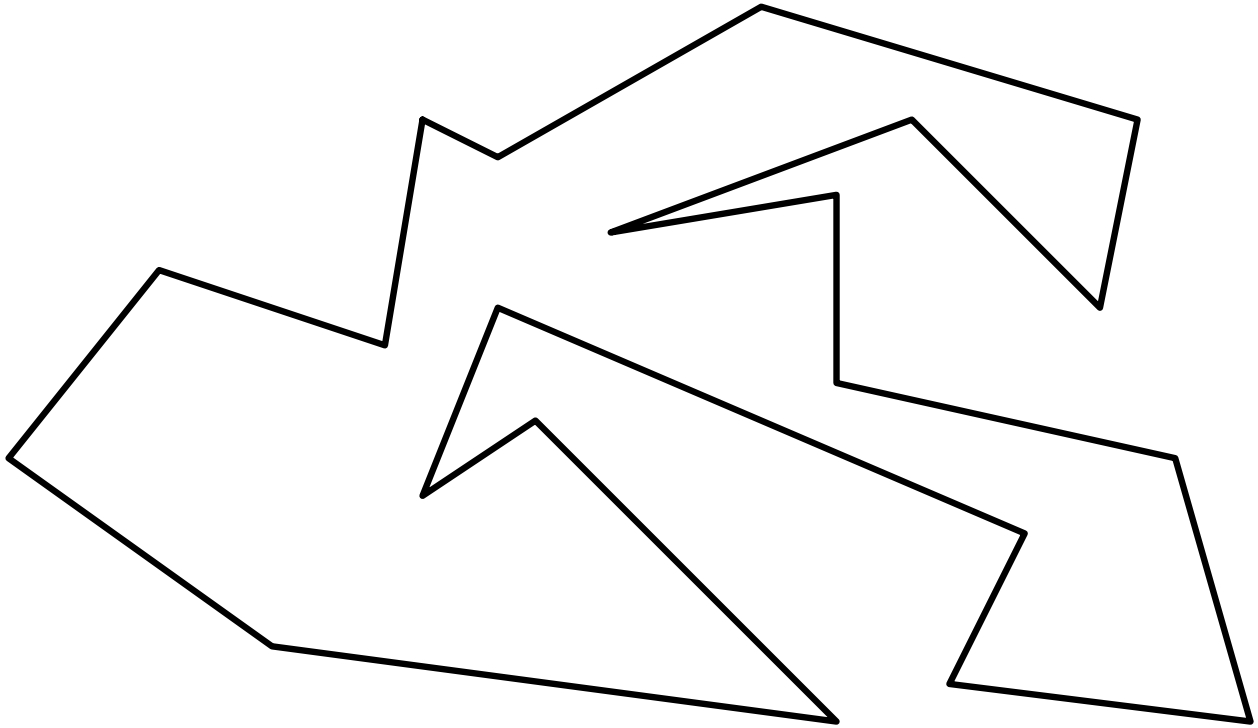
Polygon Triangulation



Mikkel Abrahamsen

Motivation: The Art Gallery Problem

Viktor Klee 1973: How many guards are needed to cover a given art gallery (polygon) with n vertices?



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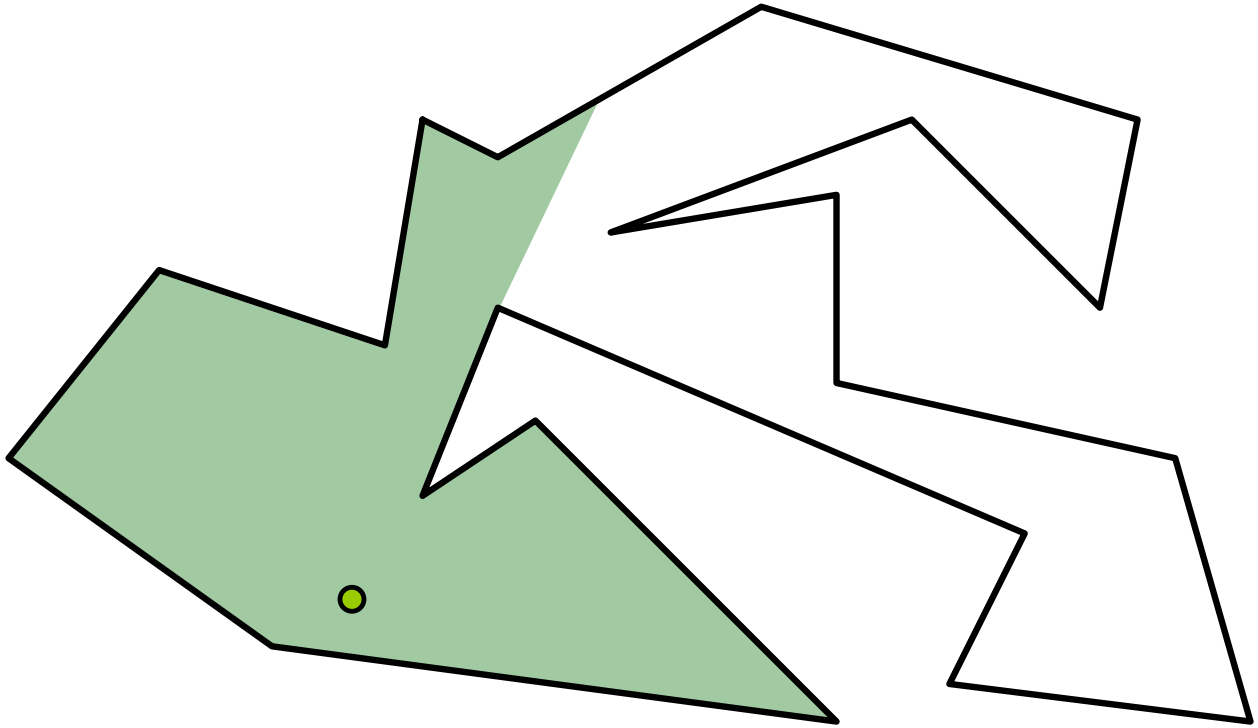

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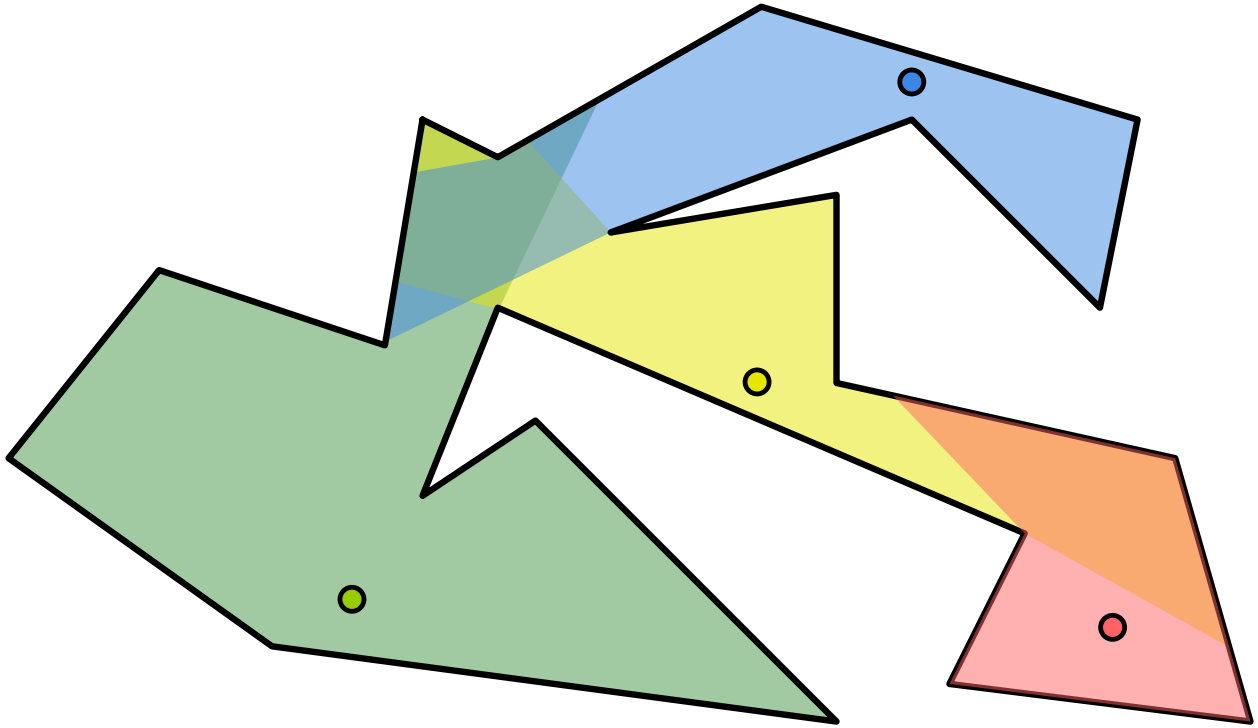

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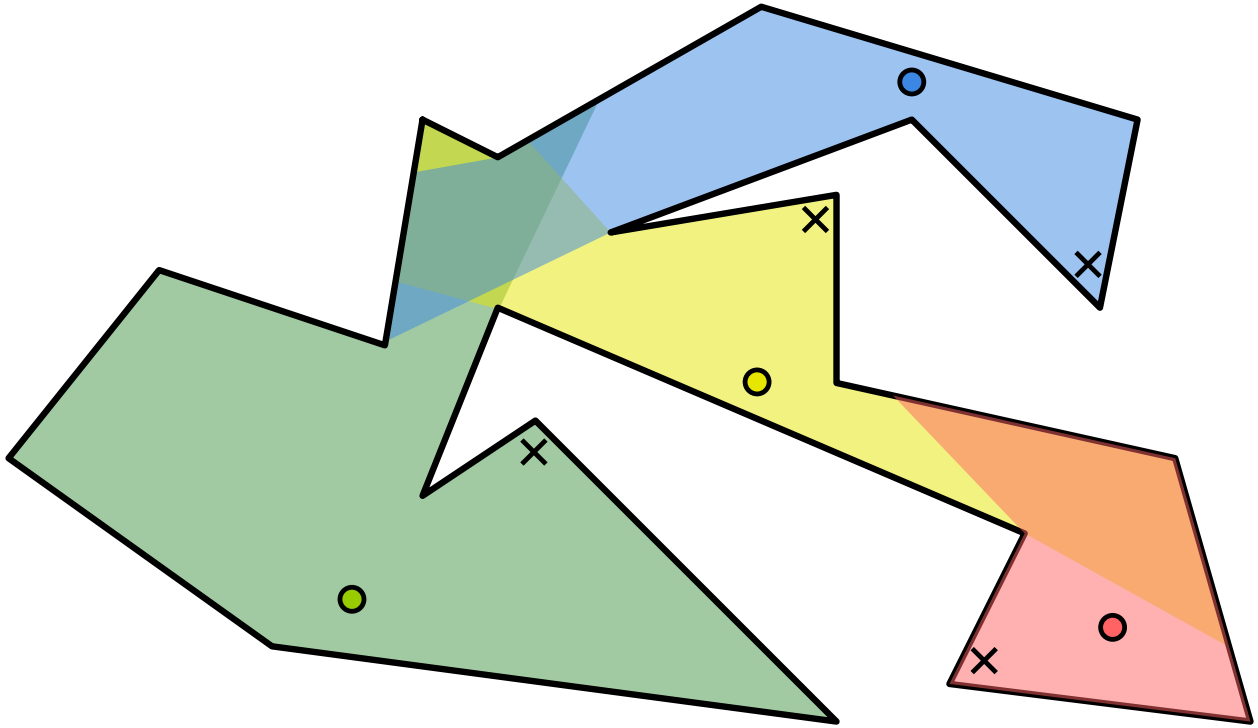

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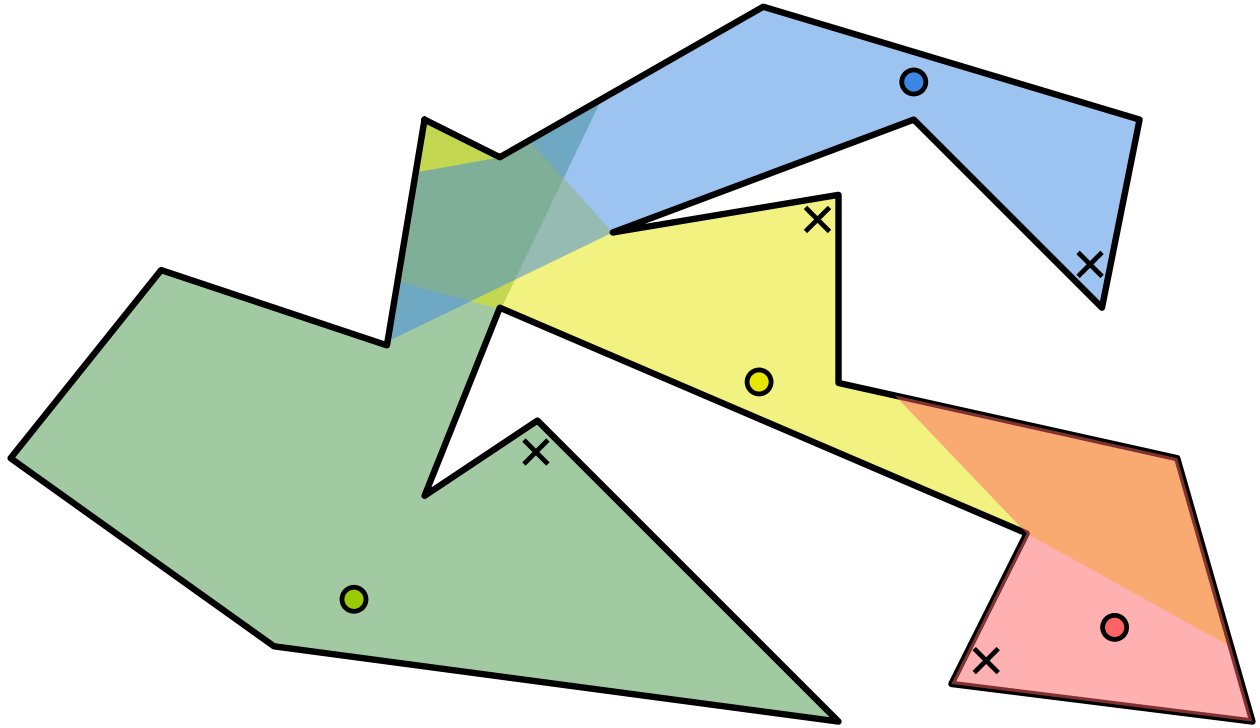

Input. Array of points $(x_1, y_1), \ldots, (x_n, y_n)$: the corners of the art gallery in cyclic order, where $x_i, y_i \in \mathbb{Z}$. An integer k.
Output. Can we place k guards that see the entire art gallery?

$\lfloor \frac{n}{3} \rfloor$ guards are sometimes needed

If $n = 3k$, then k guards can be needed!

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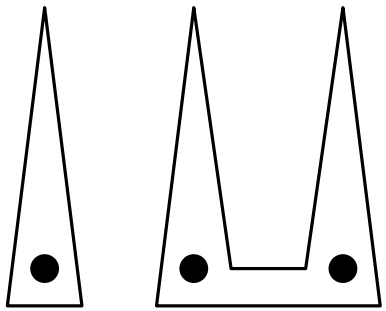

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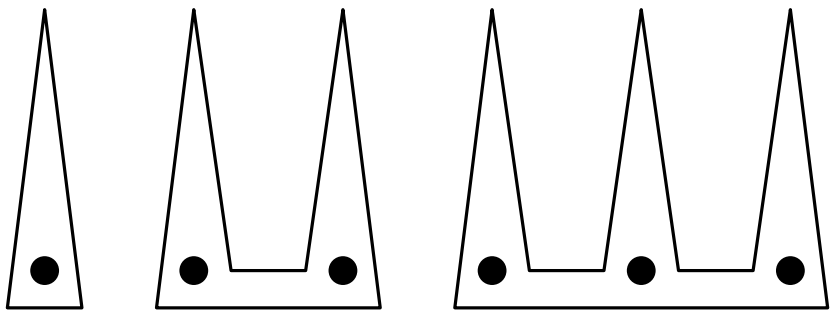

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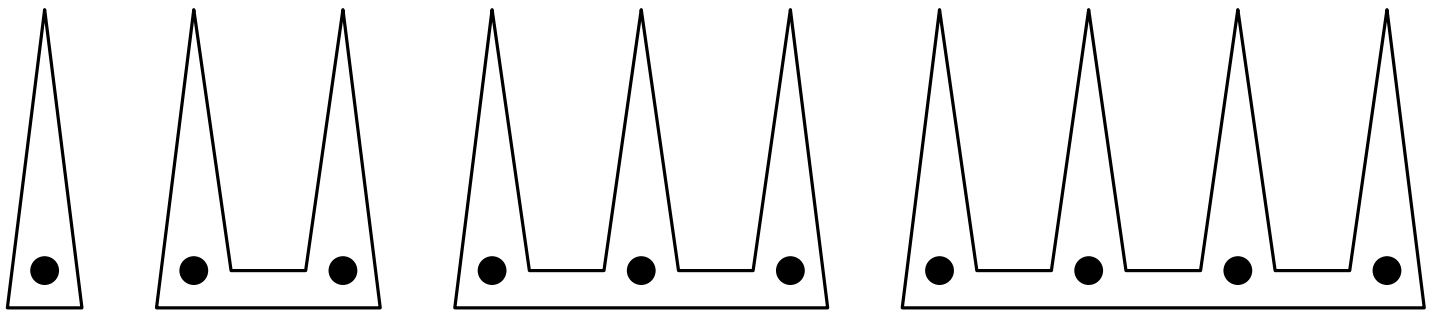

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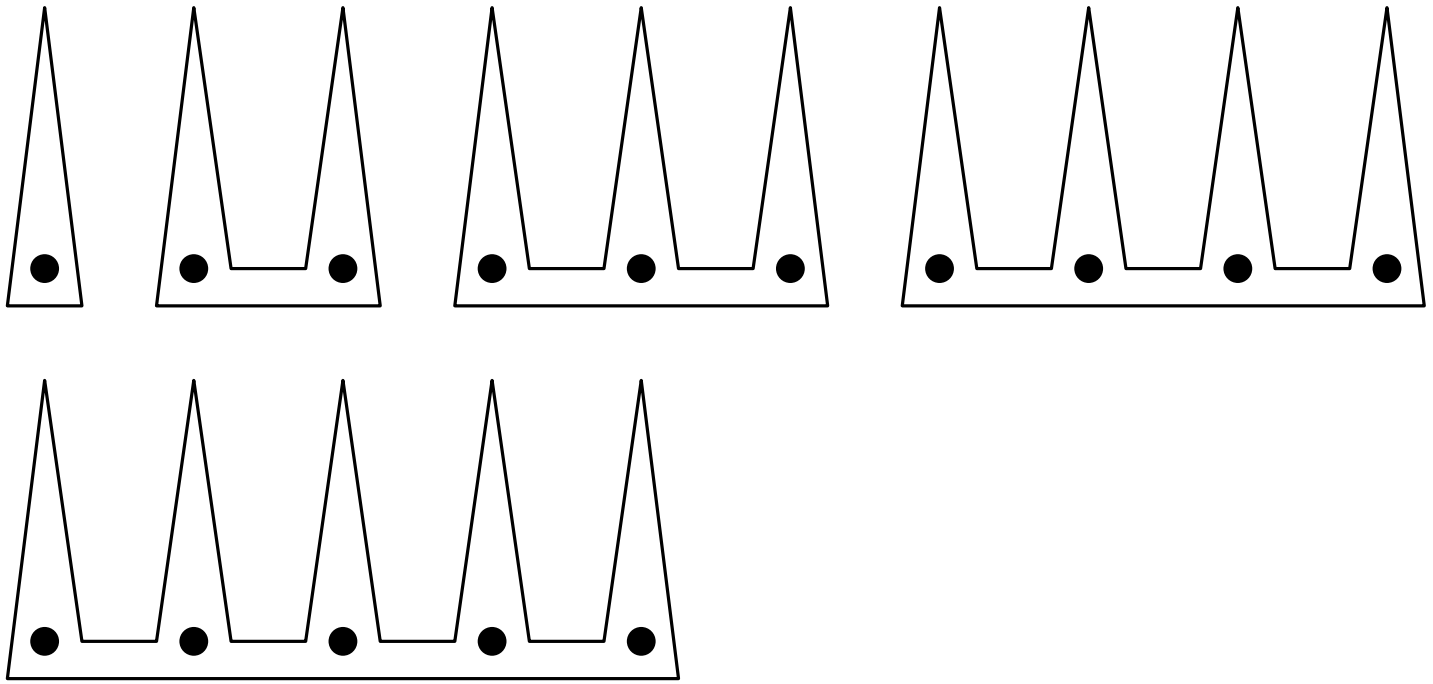

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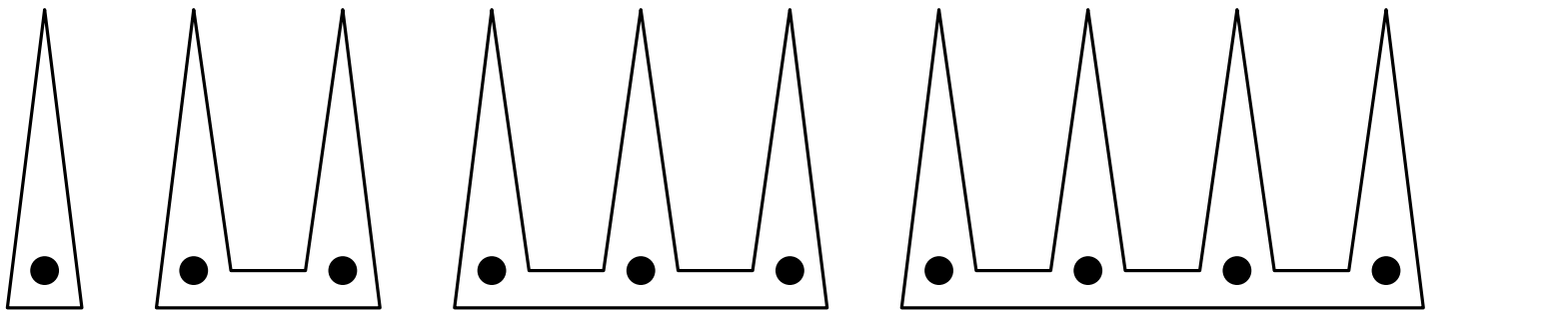

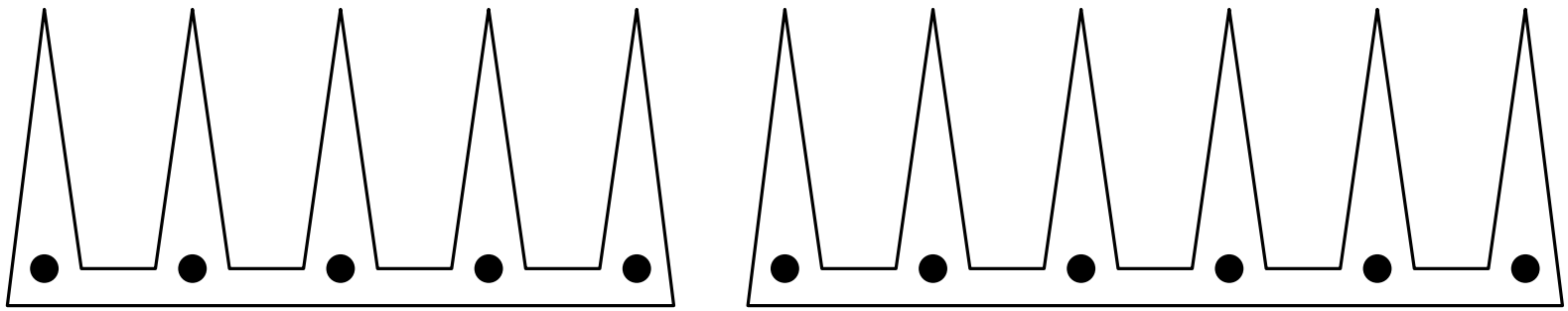

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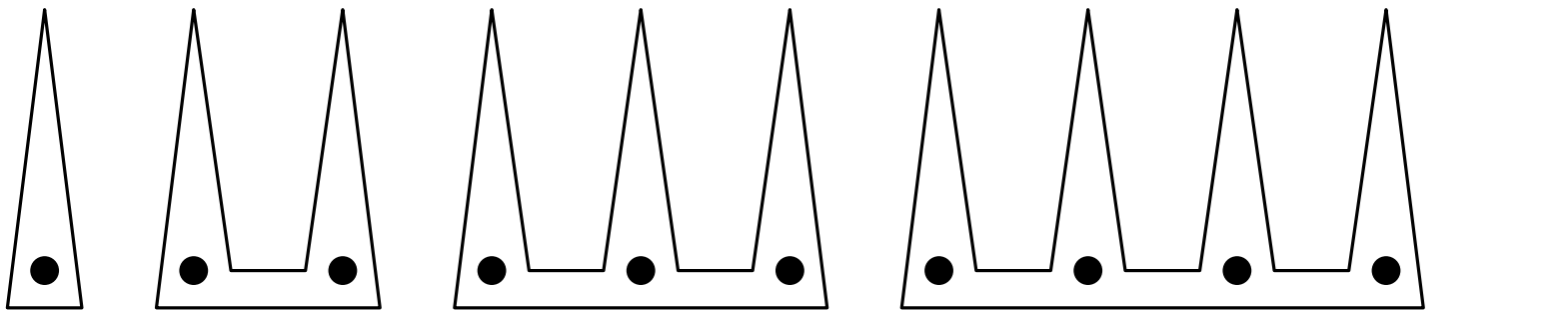
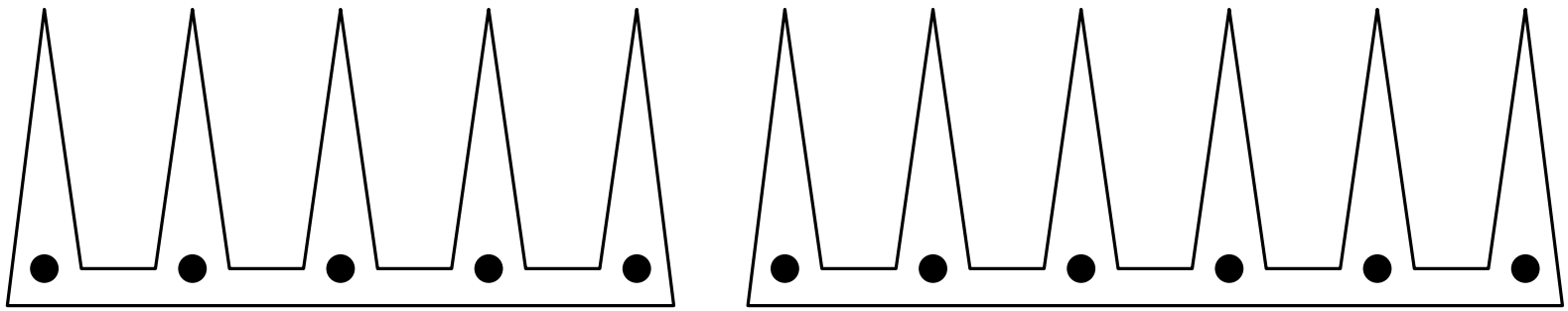
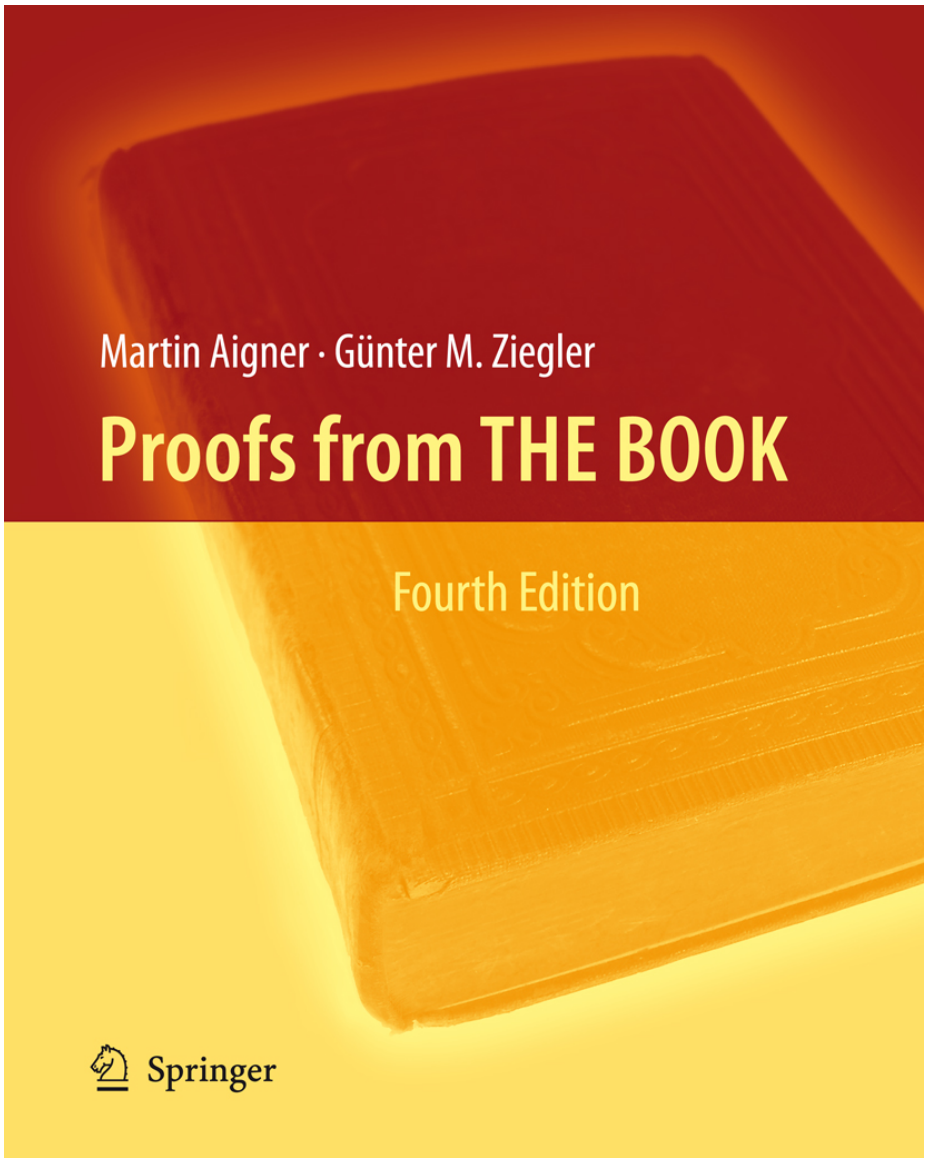


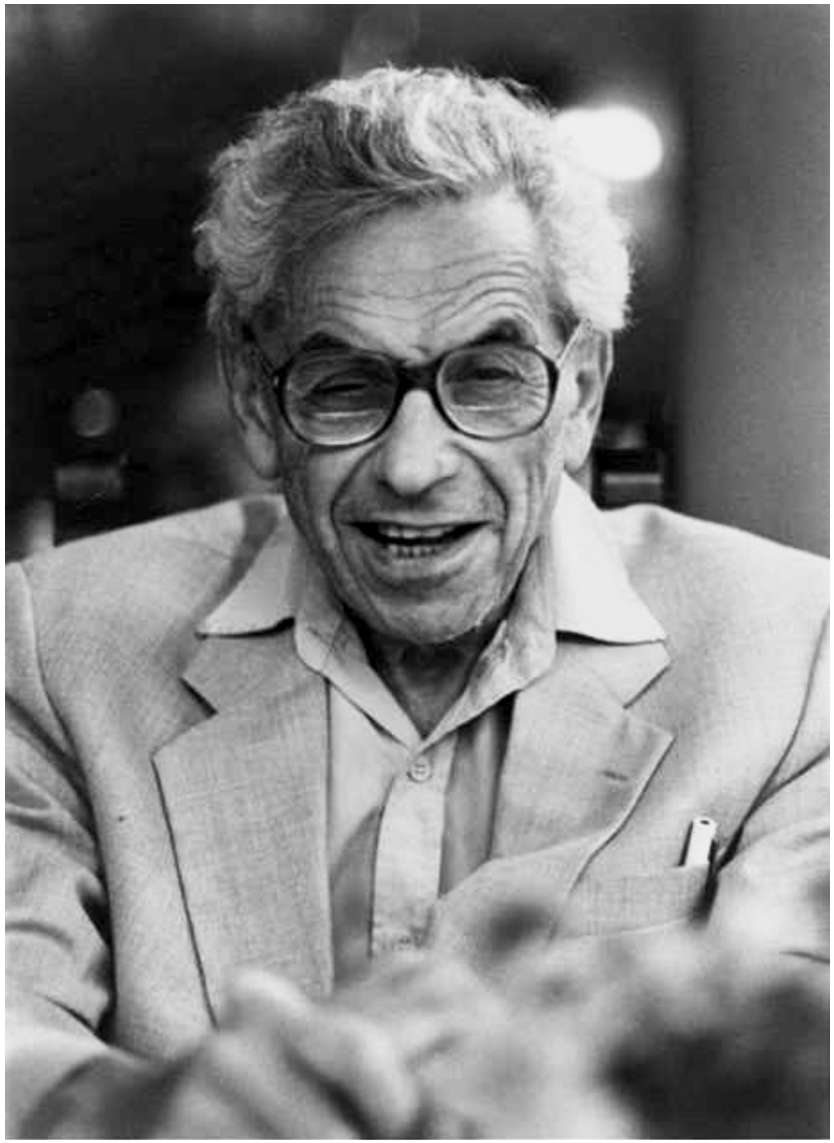


Conclusion: $\lfloor \frac{n}{3} \rfloor$ guards are sometimes needed.

Question: Is it always enough?

Proof that $\lfloor \frac{n}{3} \rfloor$ are sufficient

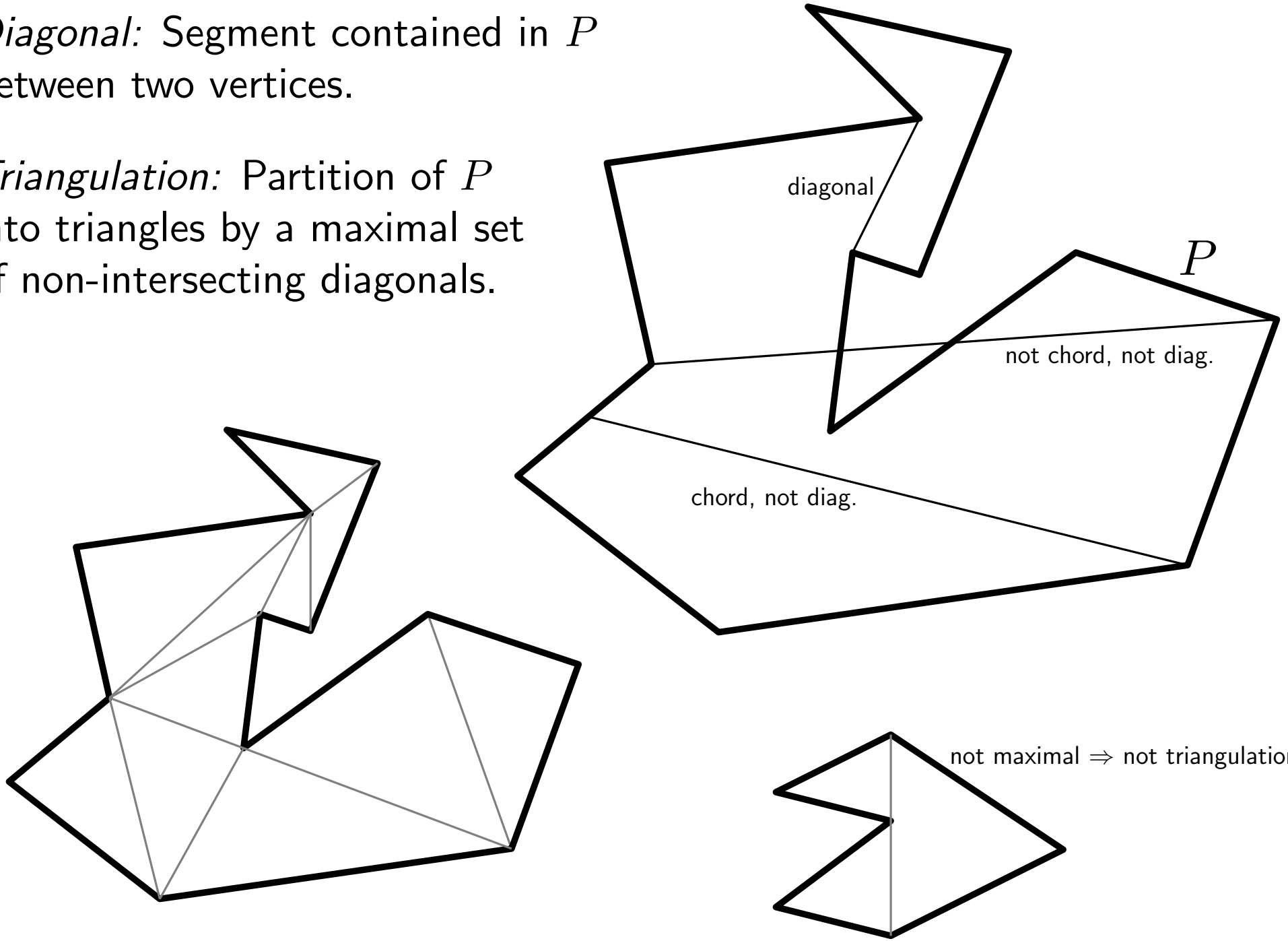




What is a triangulation?

Diagonal: Segment contained in P between two vertices.

Triangulation: Partition of P into triangles by a maximal set of non-intersecting diagonals.



Any polygon can be triangulated

Lemma: A polygon P with n vertices can be triangulated, and any triangulation has $n-2$ triangles using $n-3$ diagonals.

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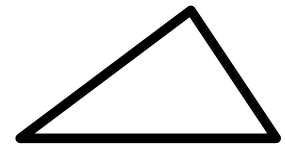

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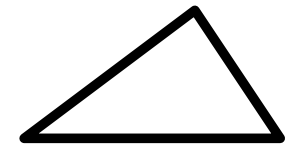

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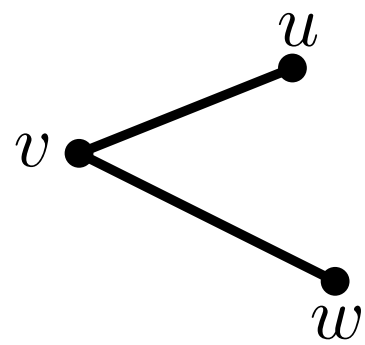

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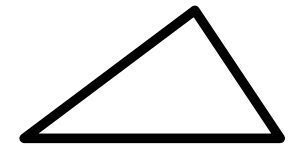

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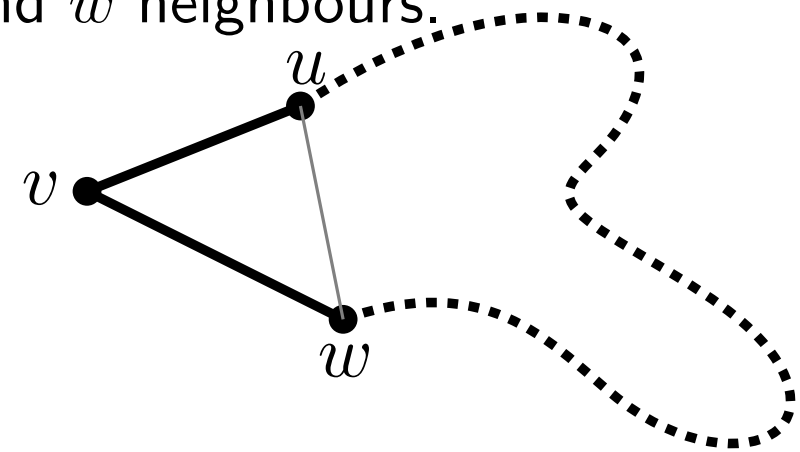

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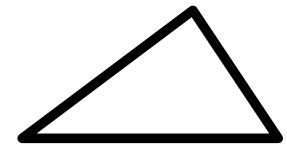

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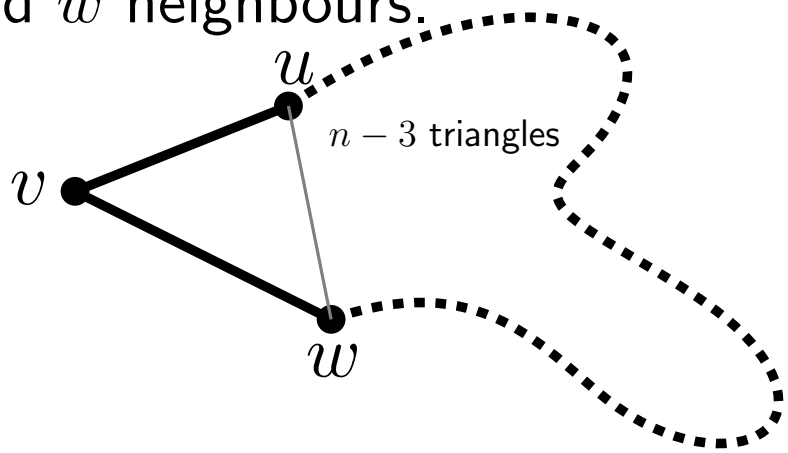
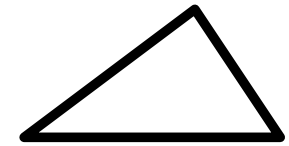
Induction hypothesis $\Rightarrow$ triangulation with $n-3$ triangles on the other side of uw.



Any polygon can be triangulated

Lemma: A polygon P with n vertices can be triangulated, and any triangulation has $n-2$ triangles using $n-3$ diagonals.



Proof: Induction on n. Base case $n = 3$ is trivial.

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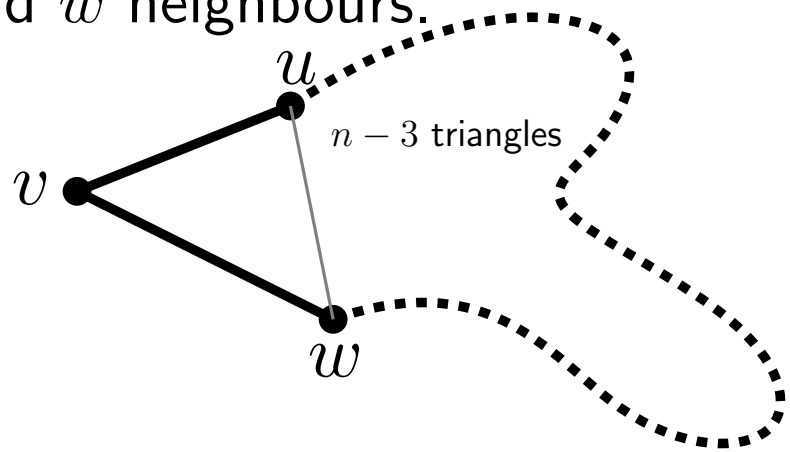
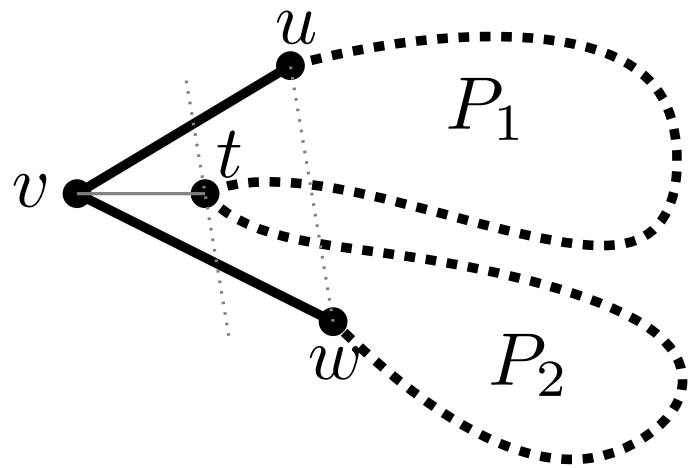


Case 2: uw is no diagonal. Let t be corner in uvw farthest from uw. vt is a diagonal, splits P into P_1 and P_2 with $m_1 < n$ and $m_2 < n$ vertices, $m_1 + m_2 = n + 2$.



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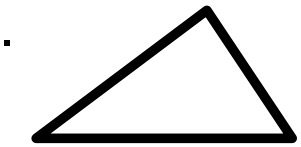

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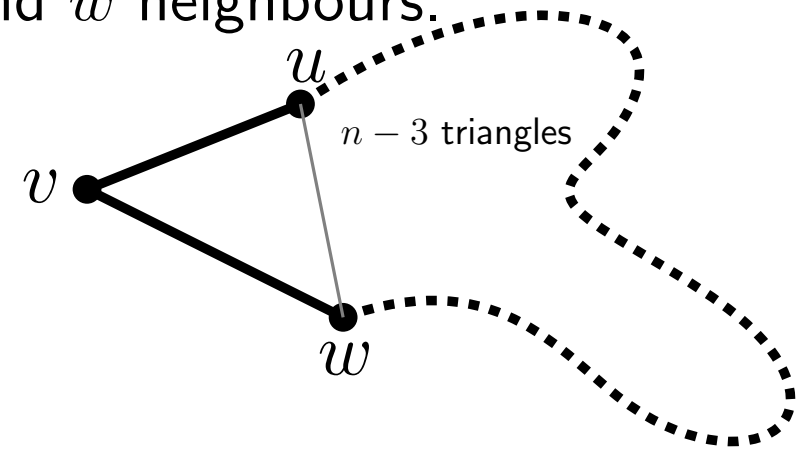

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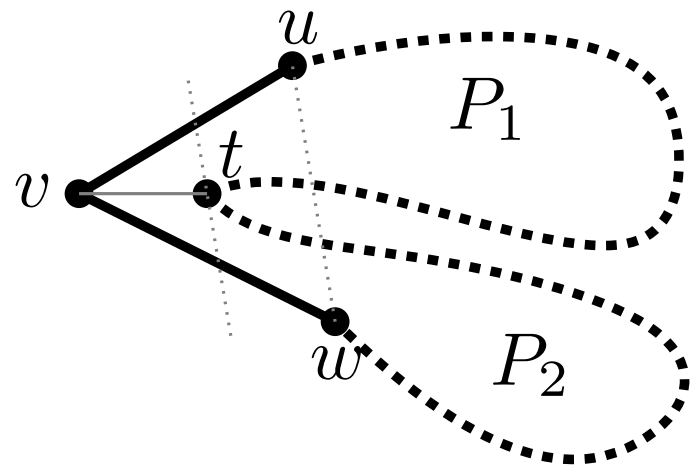
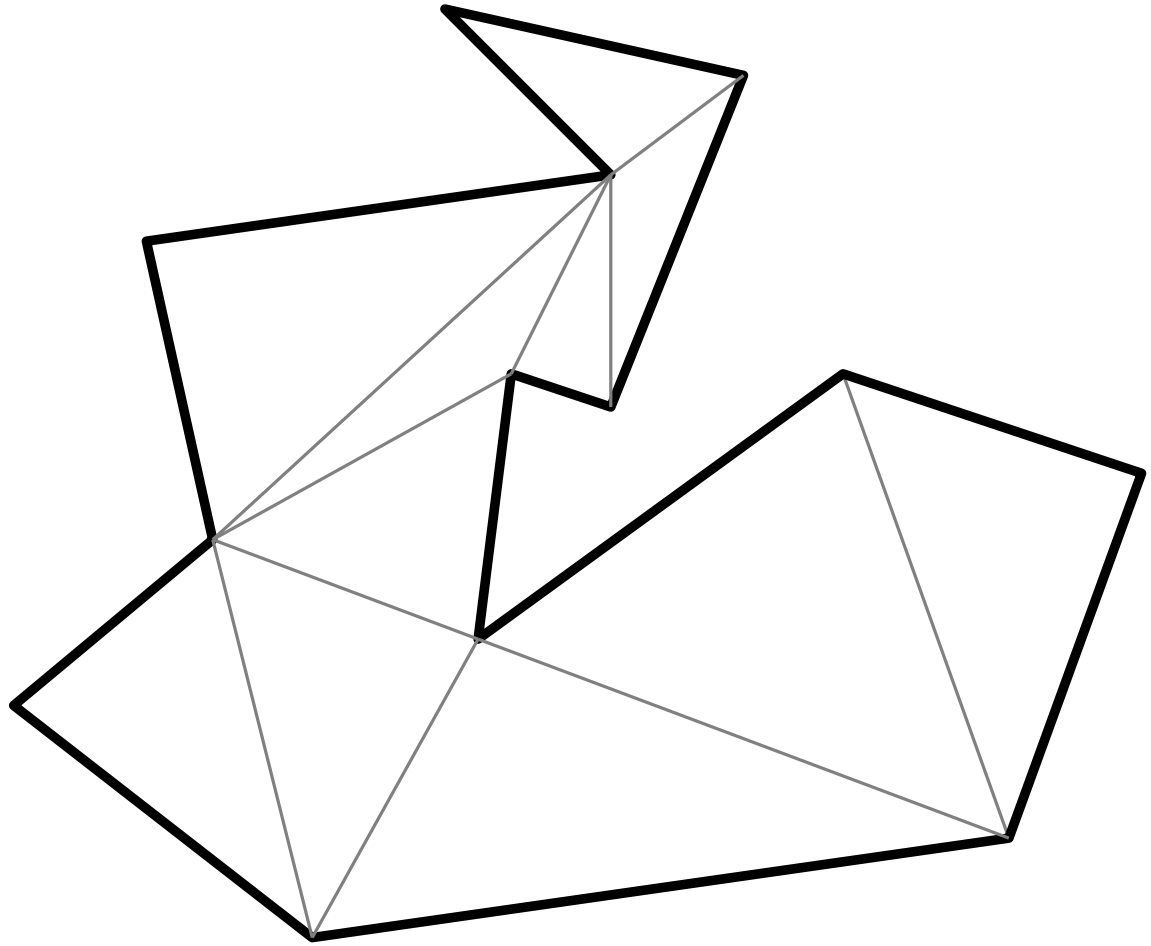


Induction hypothesis $\Rightarrow$
P_1: $m_1 - 2$ triangles.
P_2: $m_2 - 2$ triangles.
In total: $m_1 + m_2 - 4 = n + 2 - 4 = n - 2$.

Proof that $\lfloor \frac{n}{3} \rfloor$ are sufficient

Consider a triangulation.



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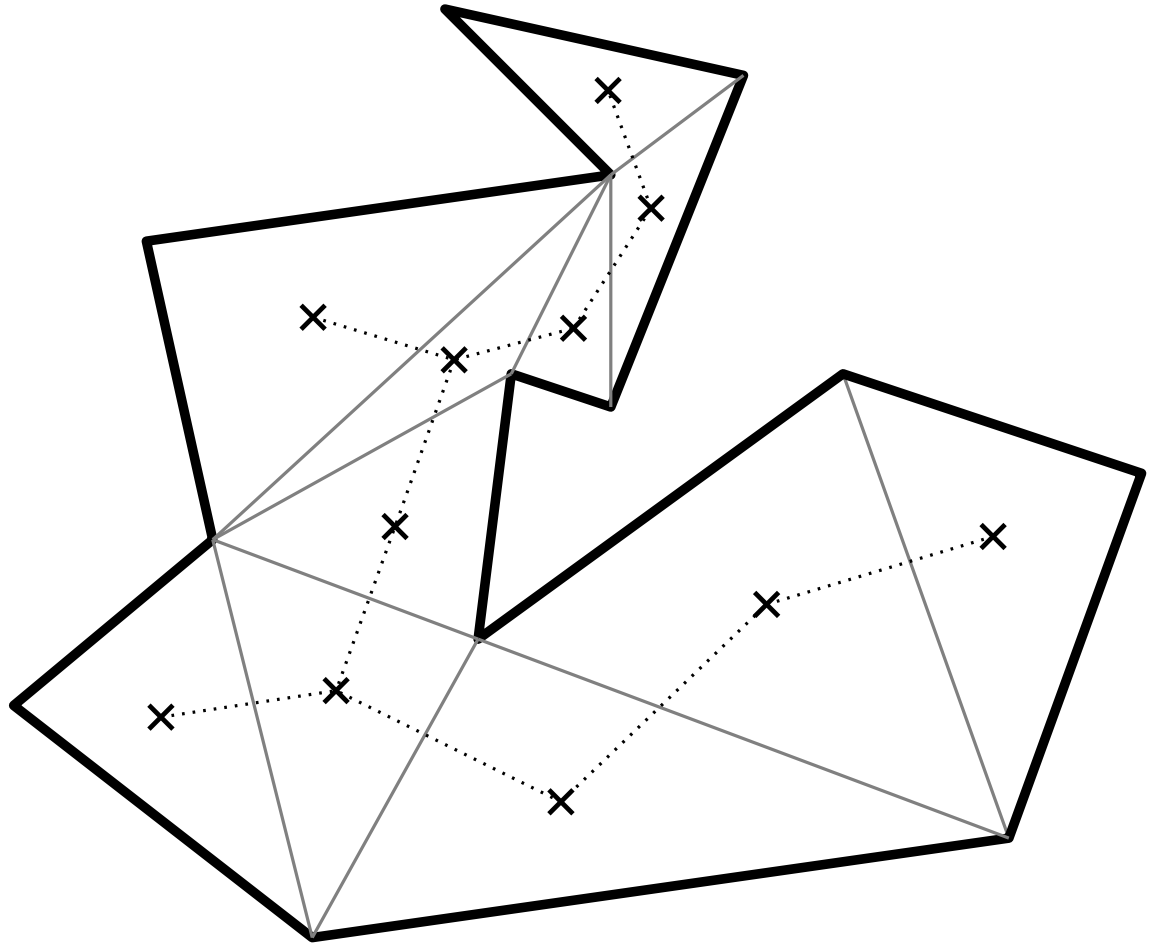

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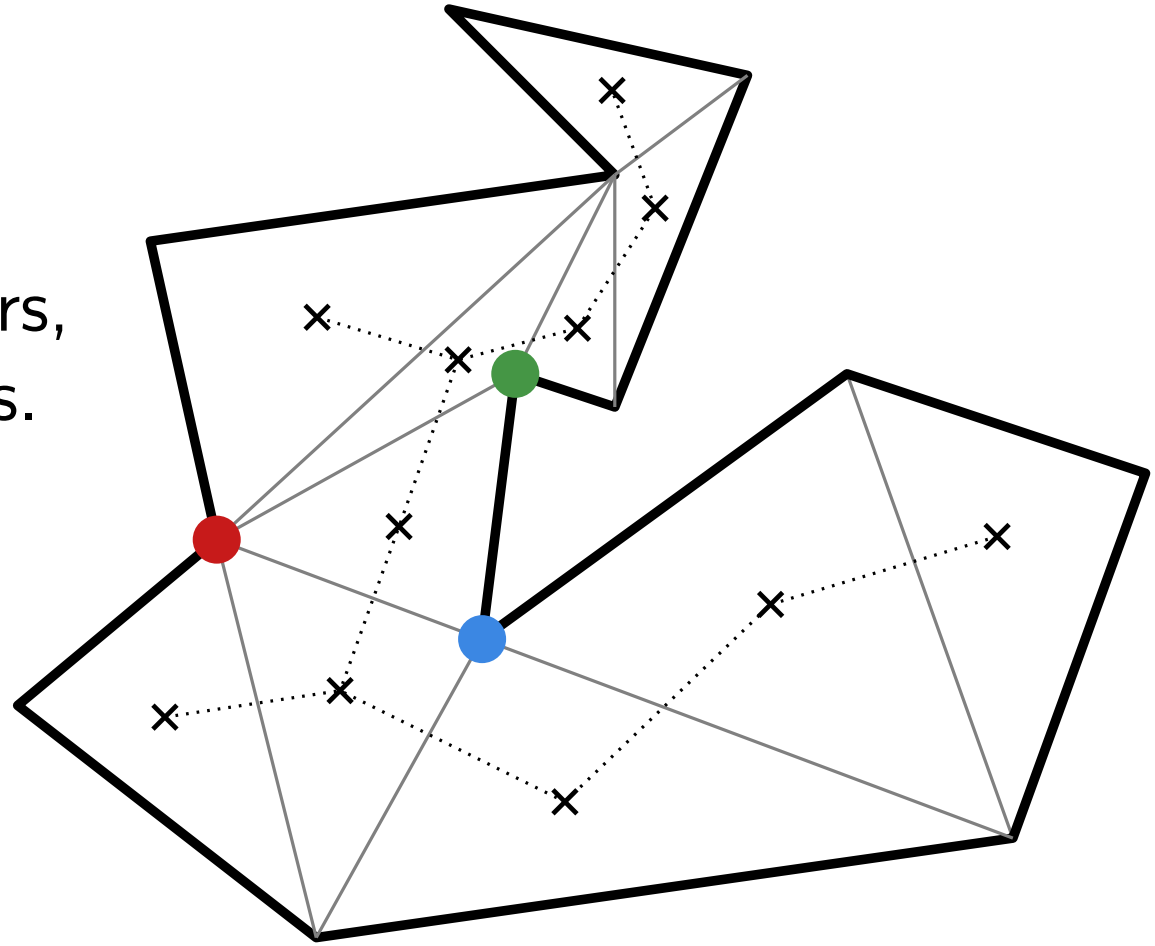
Color vertices with 3 colors, each triangle gets 3 colors.



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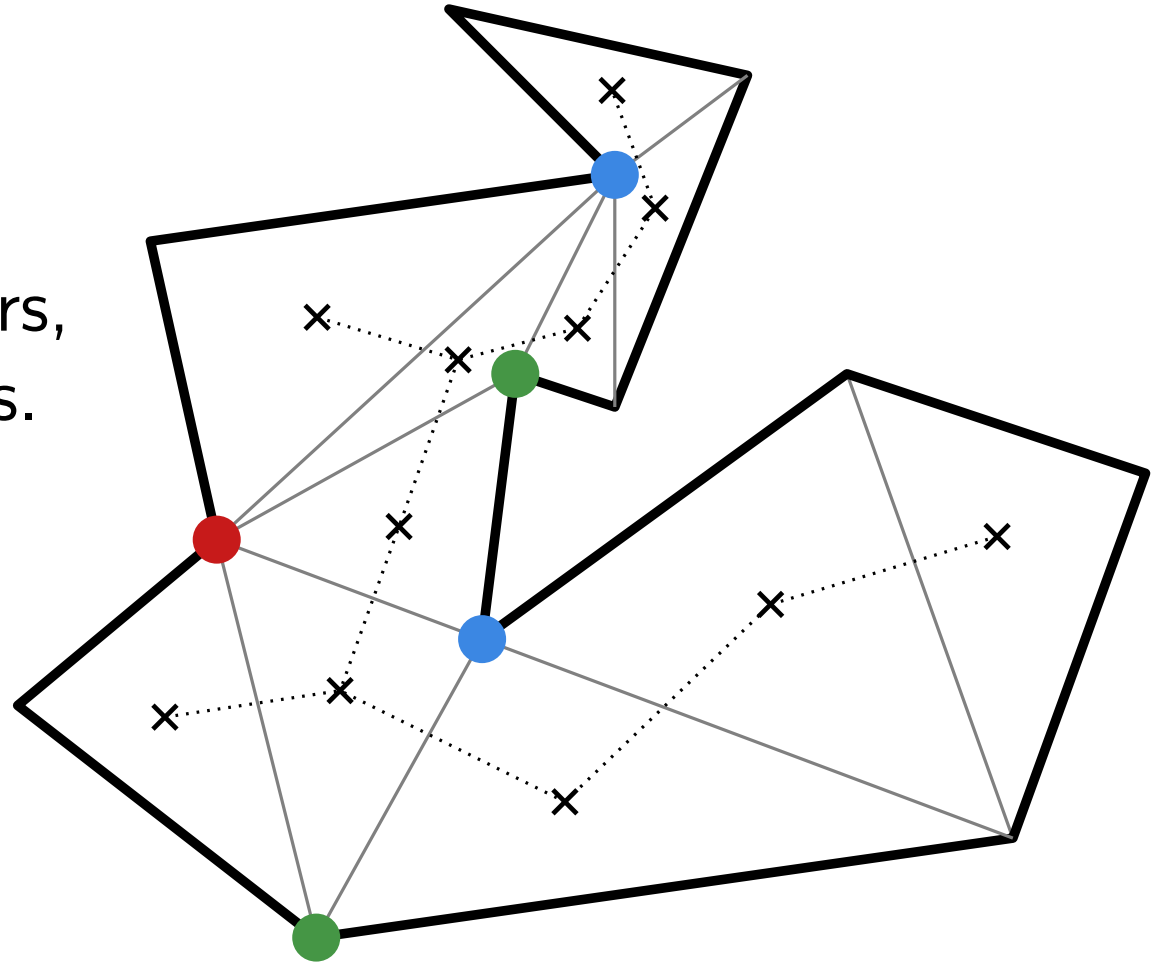

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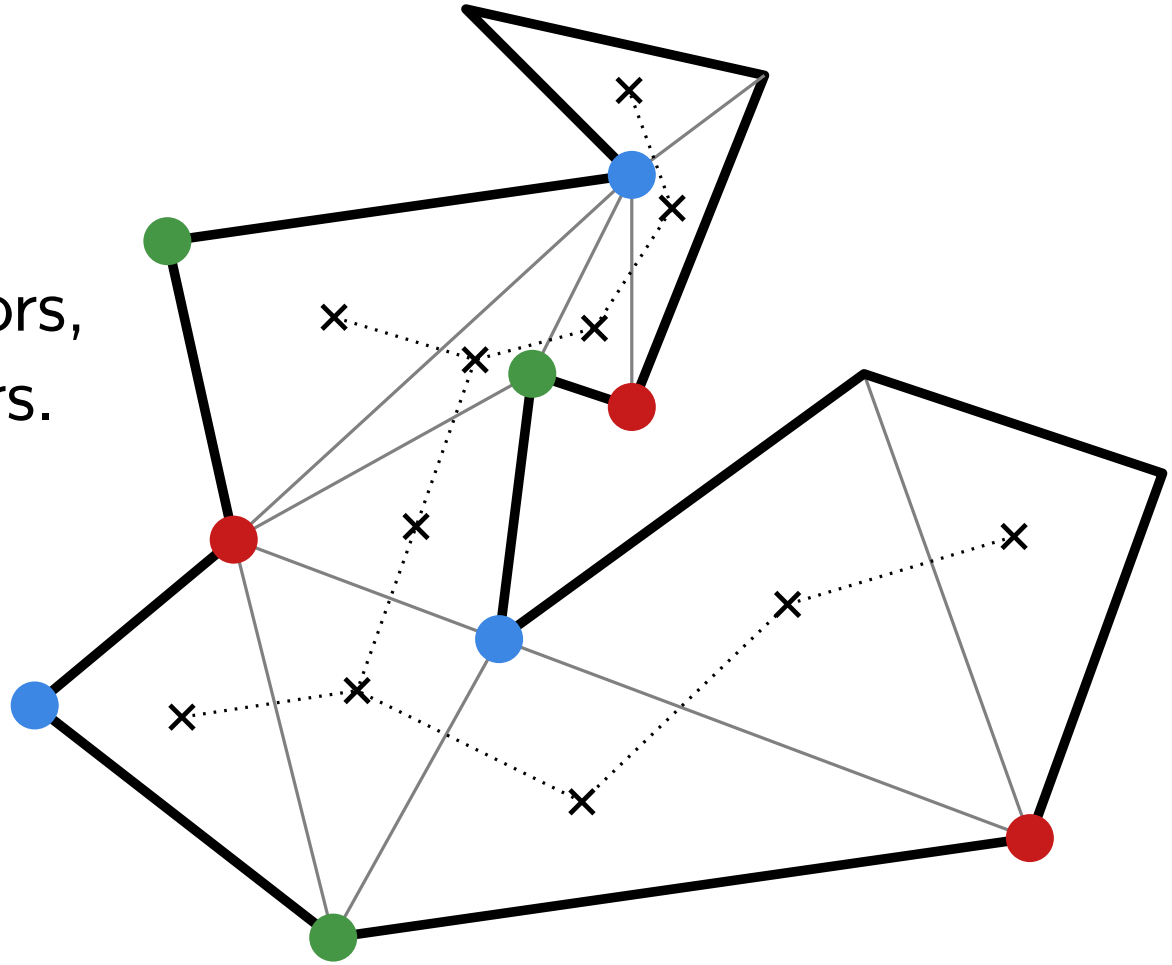

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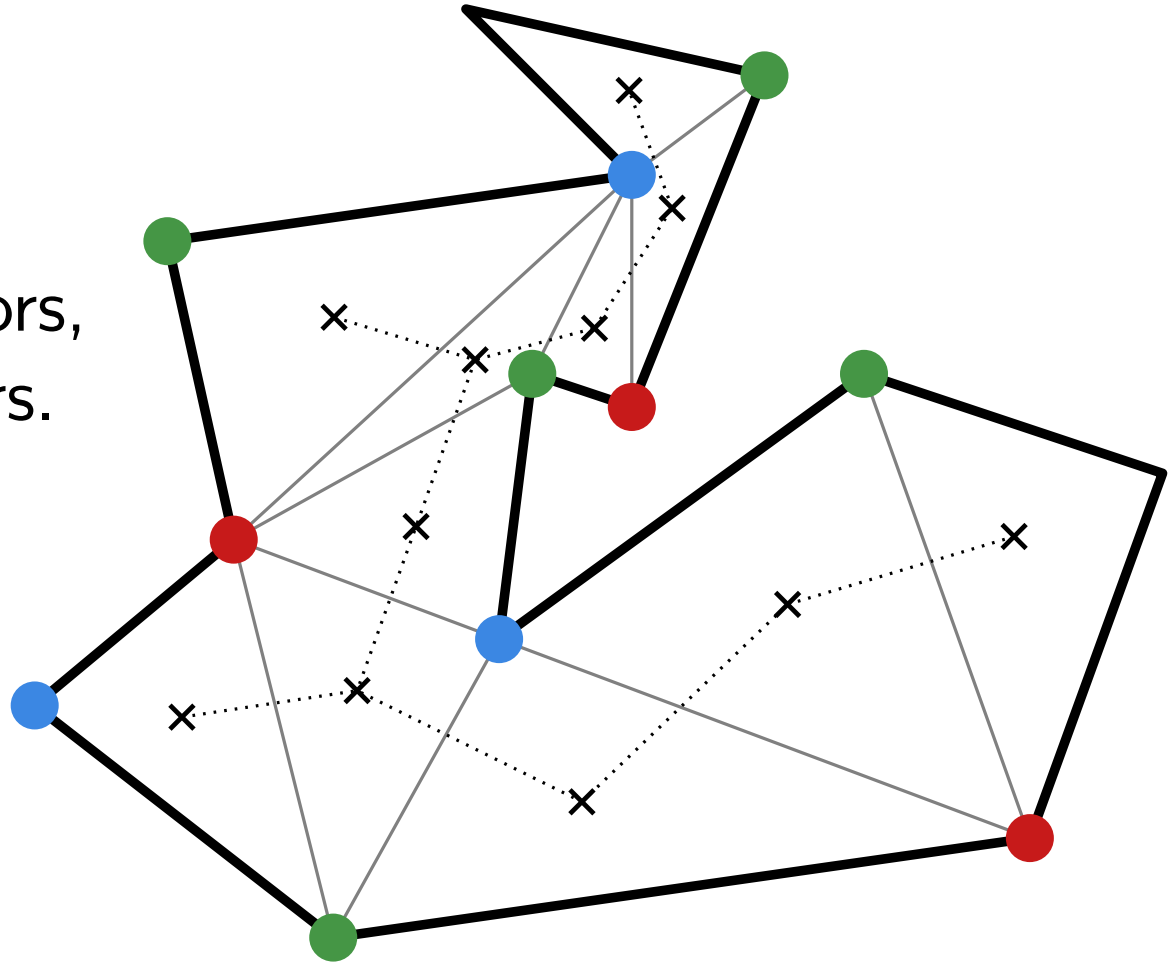

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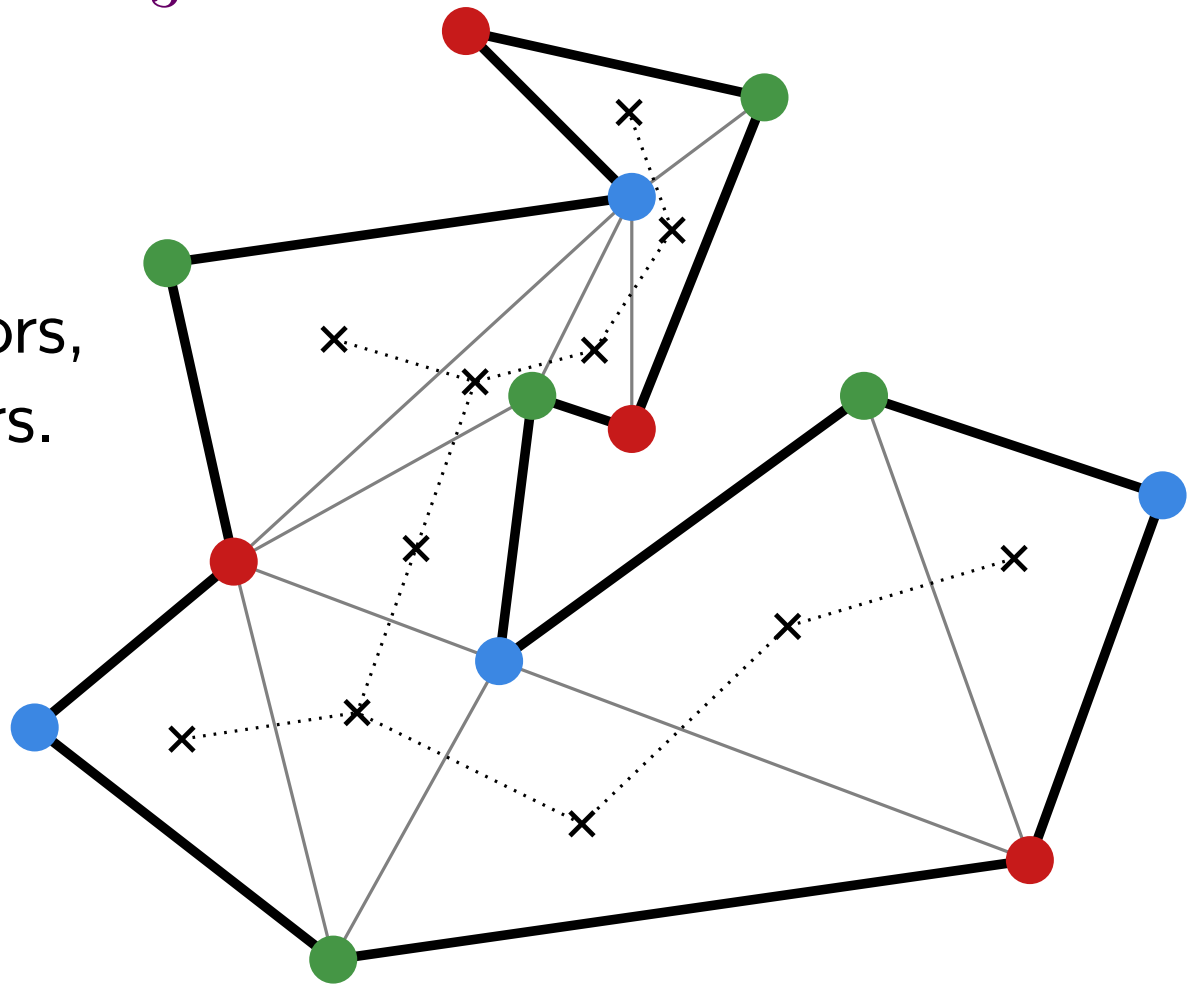

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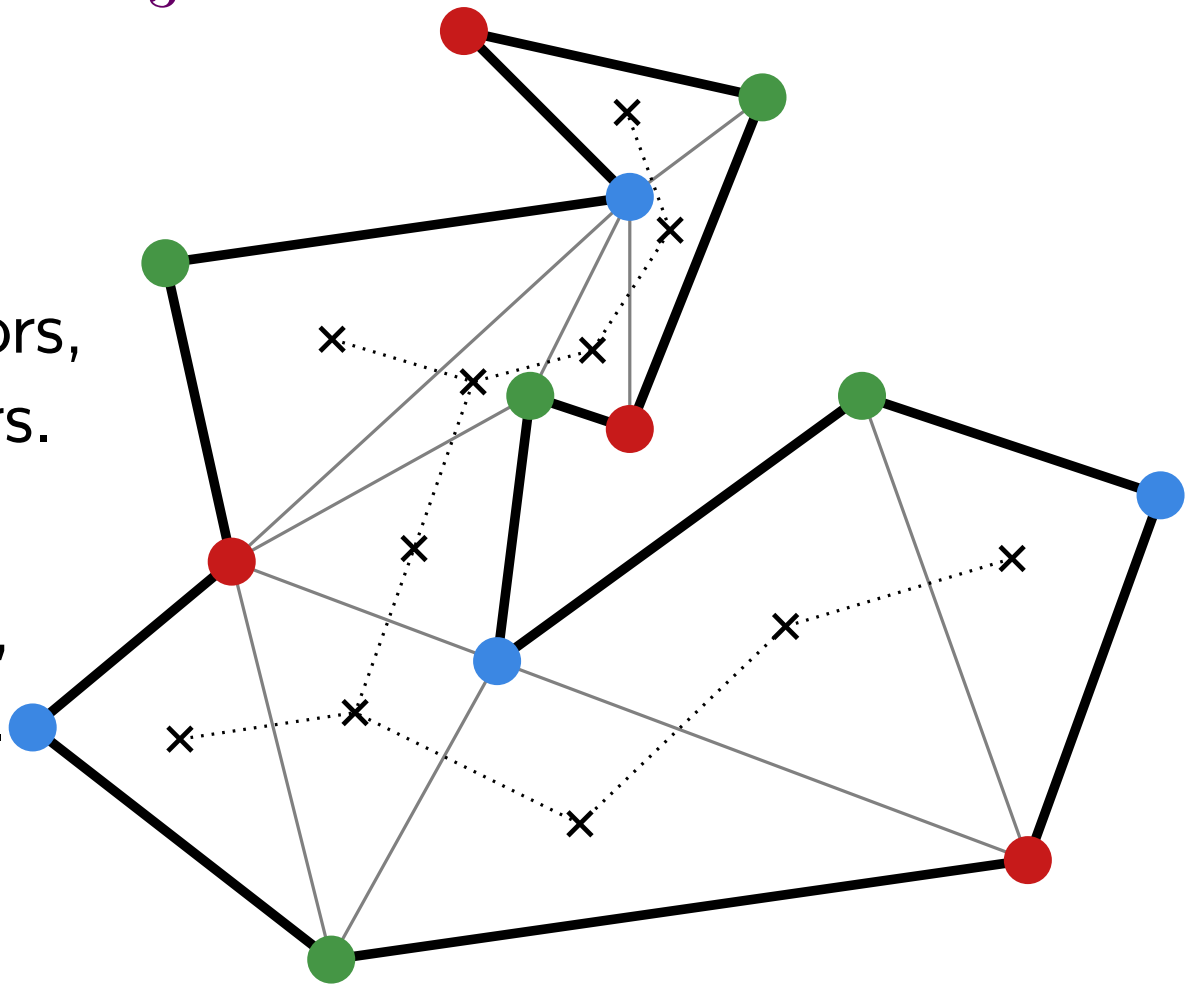
Observe that the red vertices guard the gallery, as do the green and blue.



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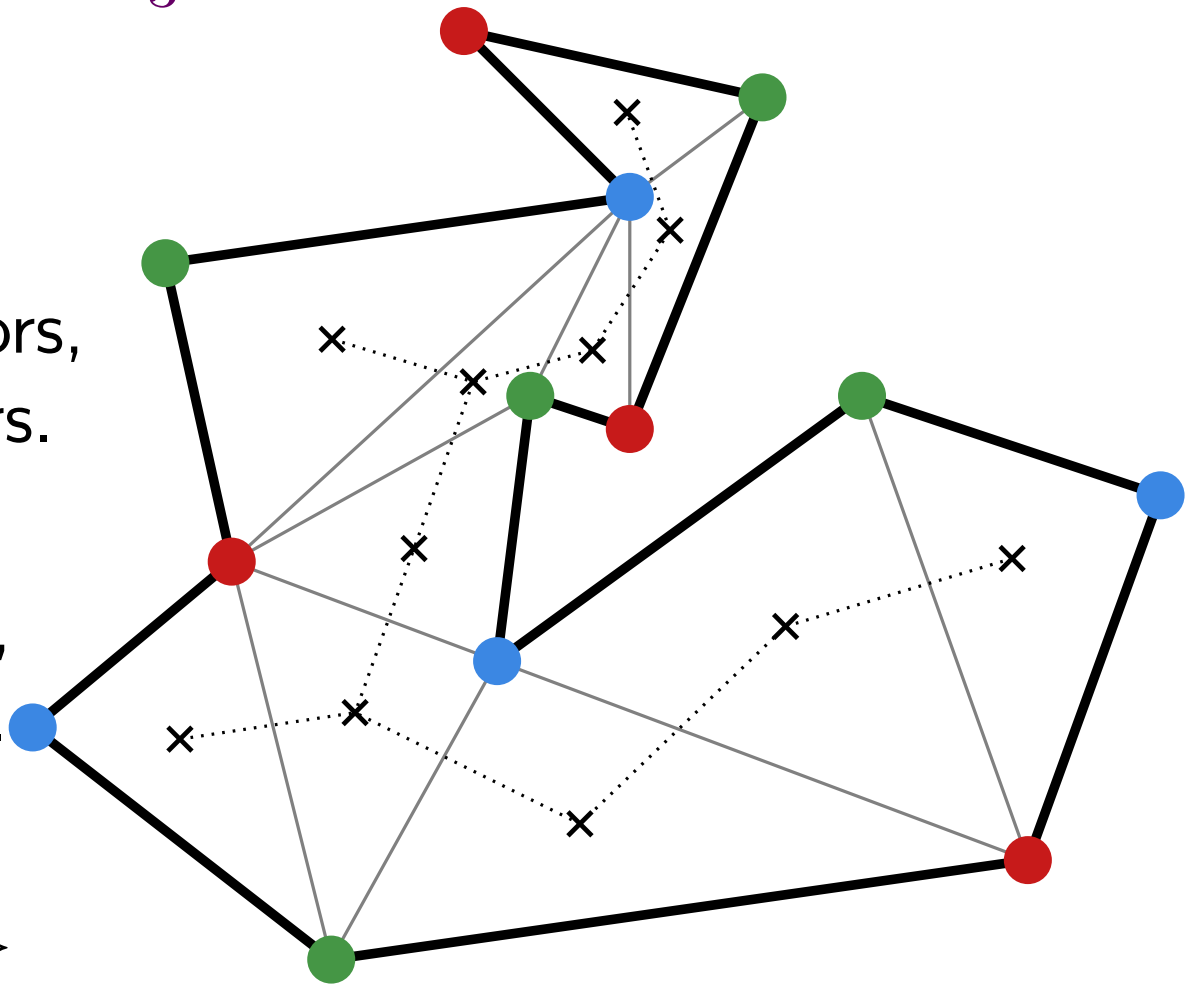

$n = n_r + n_g + n_b \Longrightarrow$
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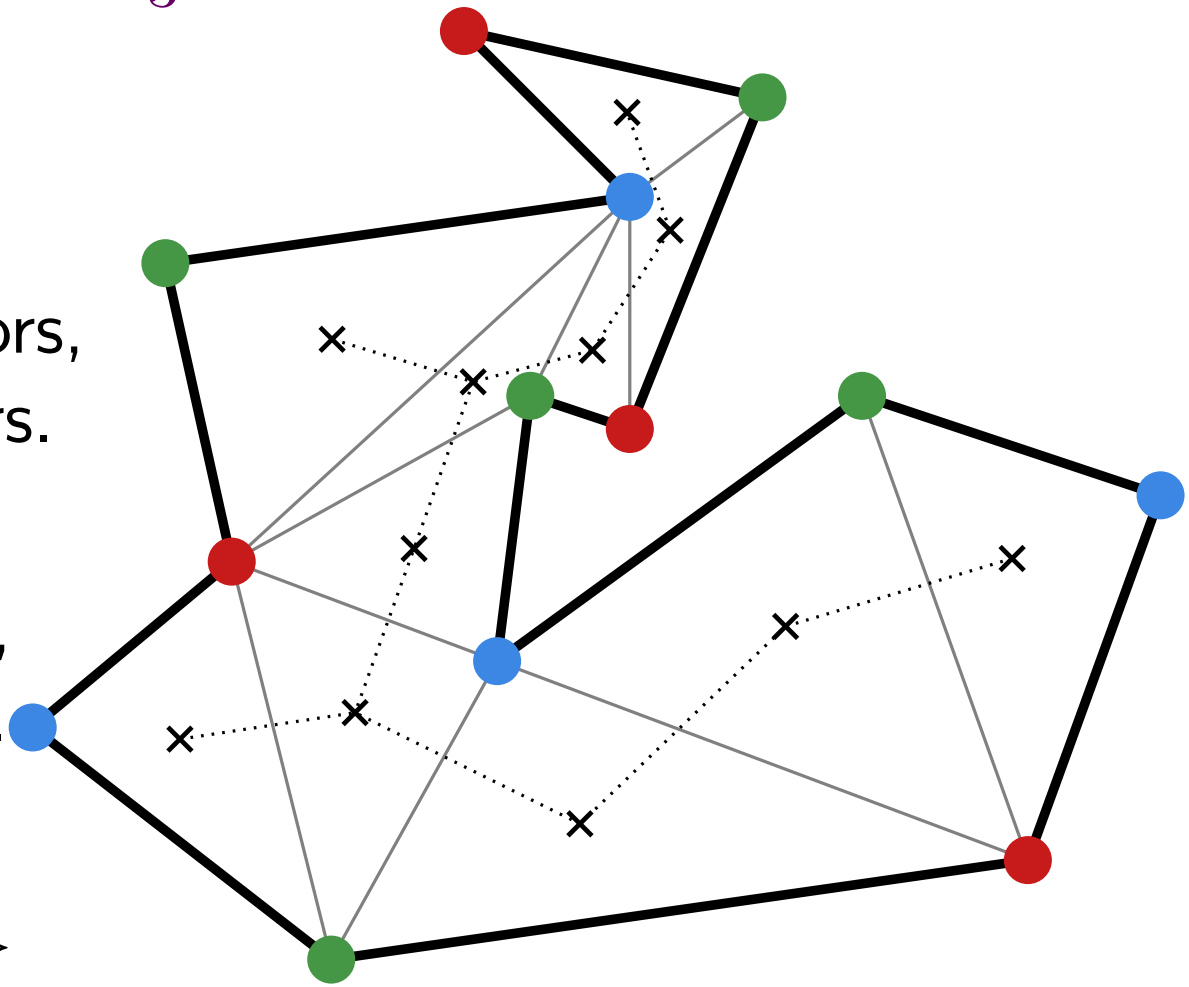

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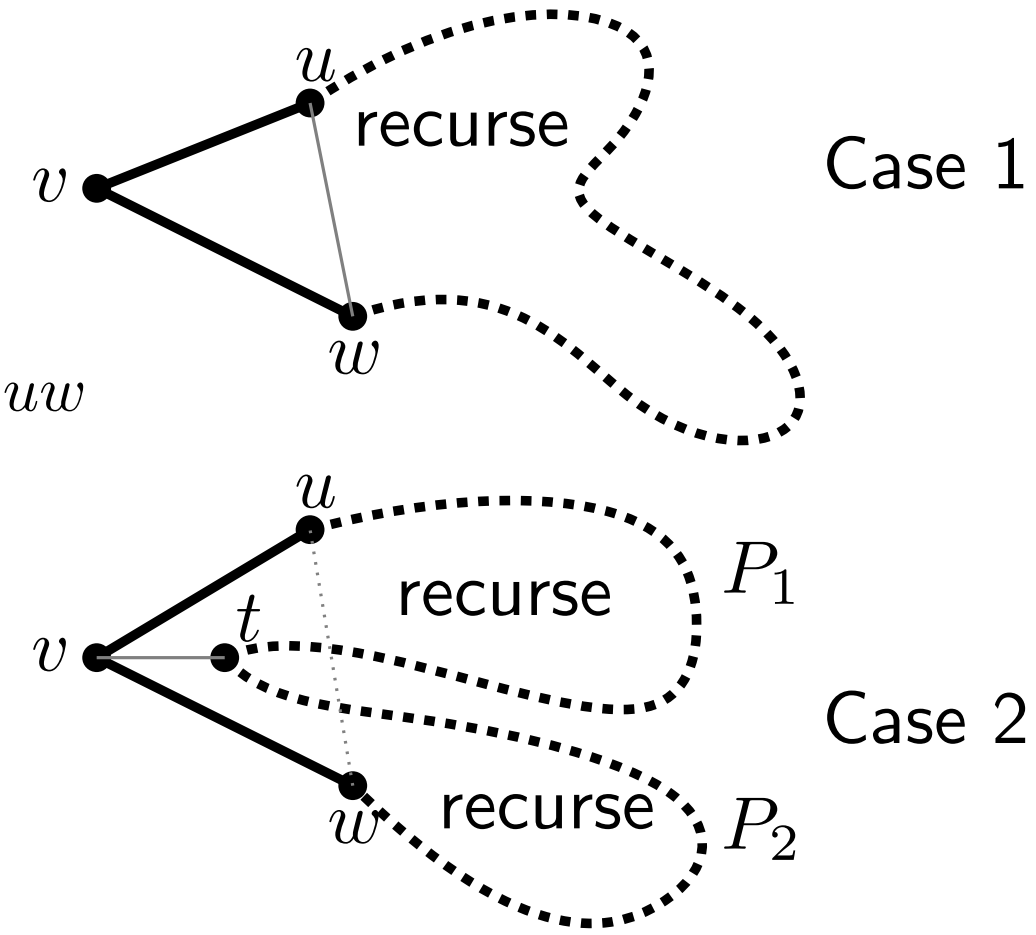
Why does the proof fail if P has holes?

Simple triangulation algorithm

Use proof that a triangulation exists!

triangulate polygon P

1. find points u, v, w
2. if uw is a diagonal // case 1
3. add diagonal uw
4. recurse on the other side of uw
5. else // case 2
6. find point t
7. add diagonal vt
8. recurse on P_1 and P_2



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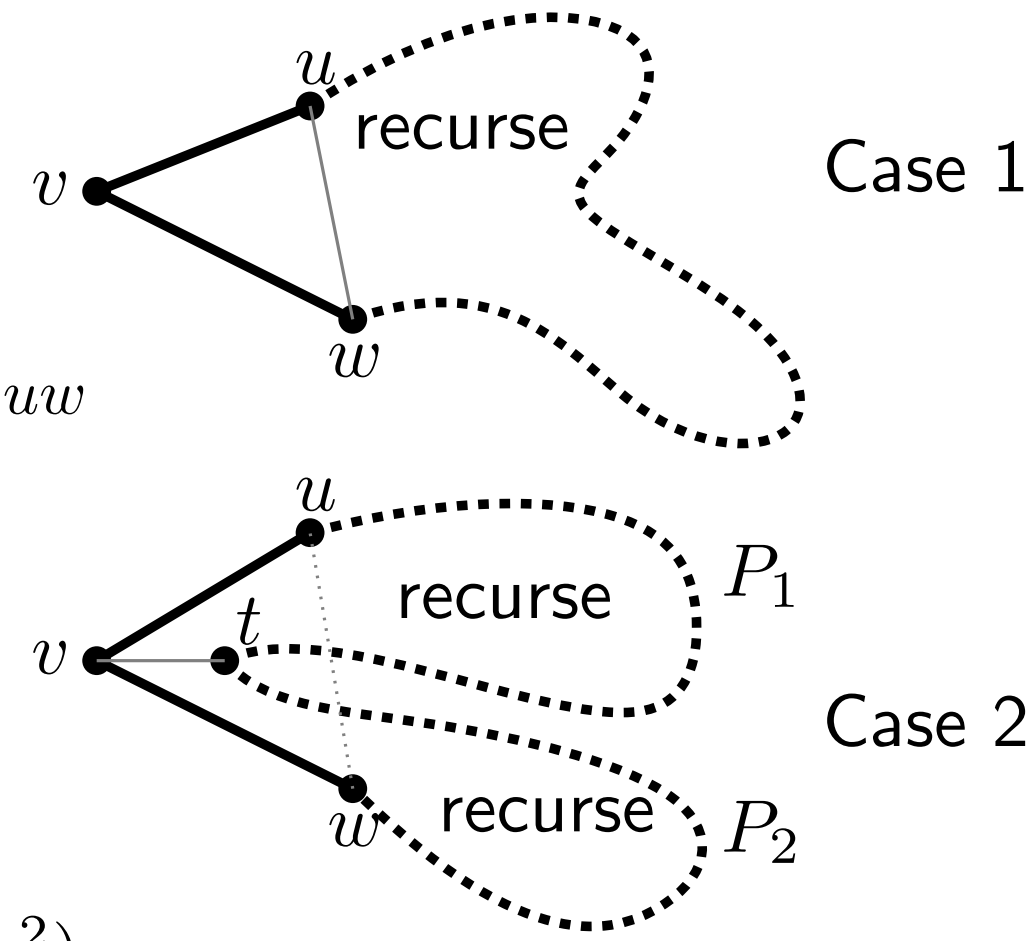
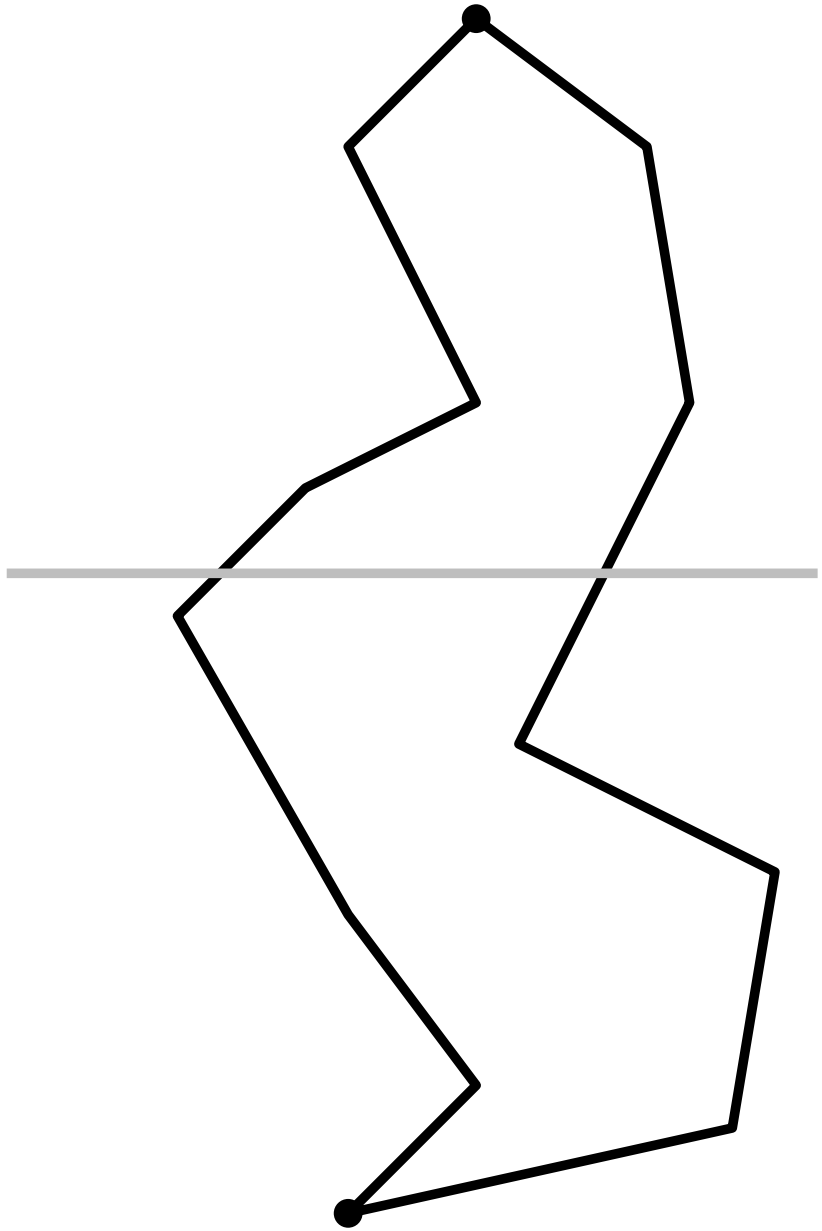


Worst-case running time: $O(n^2)$.

Faster algorithm

y-monotone polygon: Intersection with any horizontal line is connected.
Equivalent: One vertex with both edges going down and one with both edges going up.



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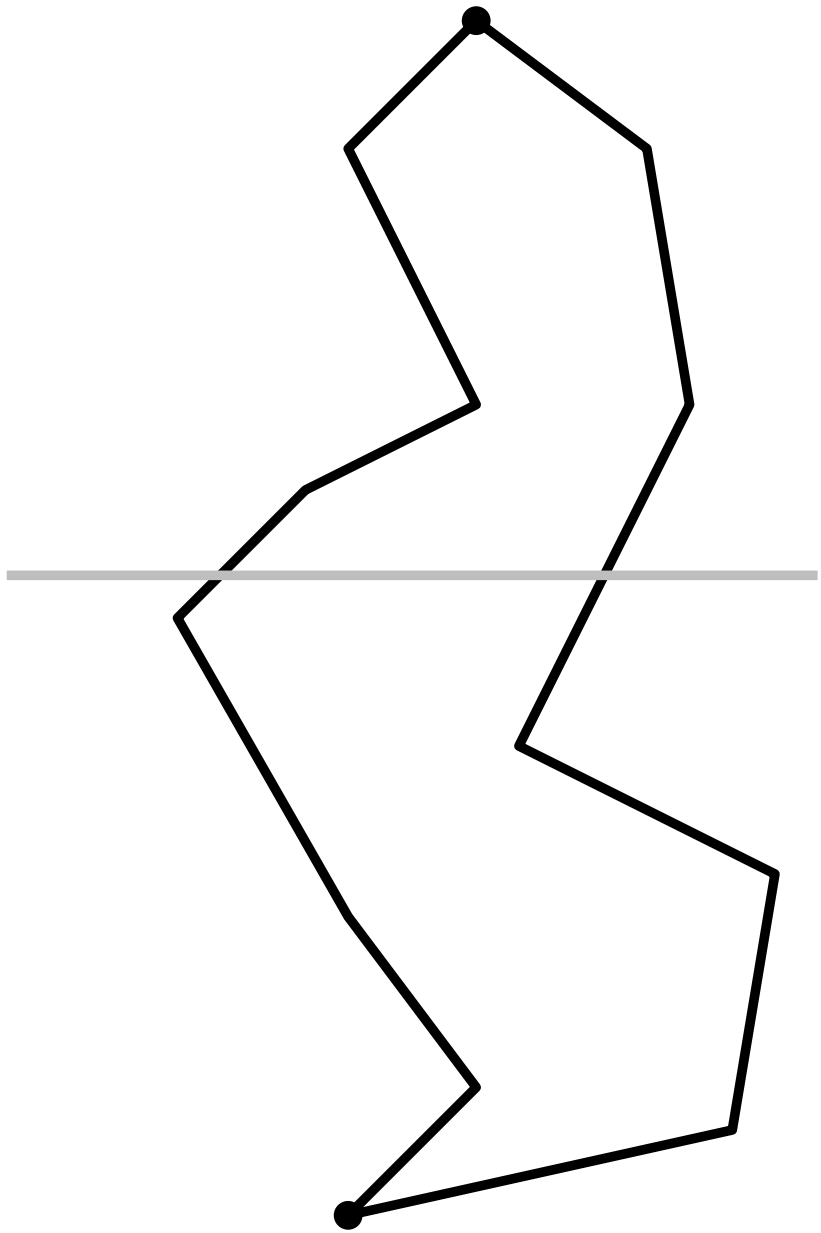
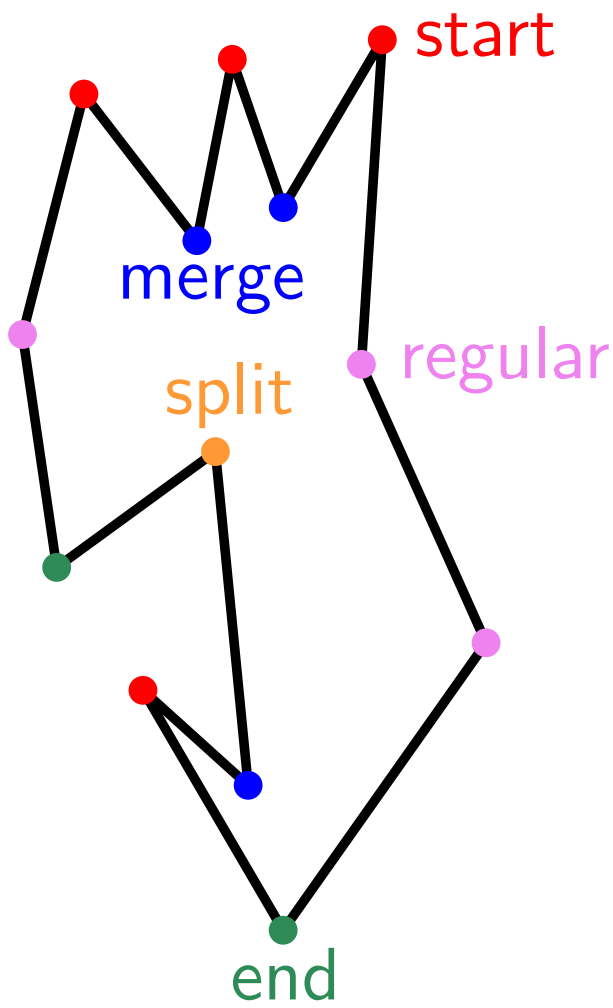


Idea:
Phase 1: Split polygon P into y-monotone polygons.
Phase 2: Triangulate each y-monotone polygon.

4 types of vertices



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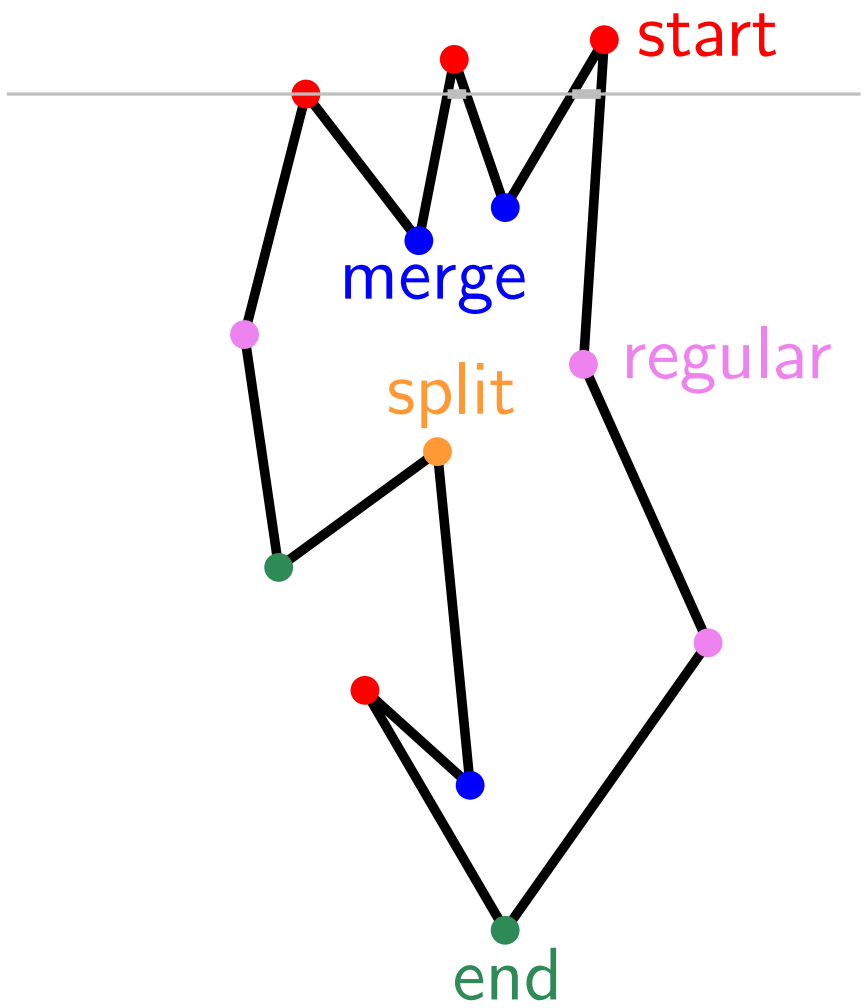

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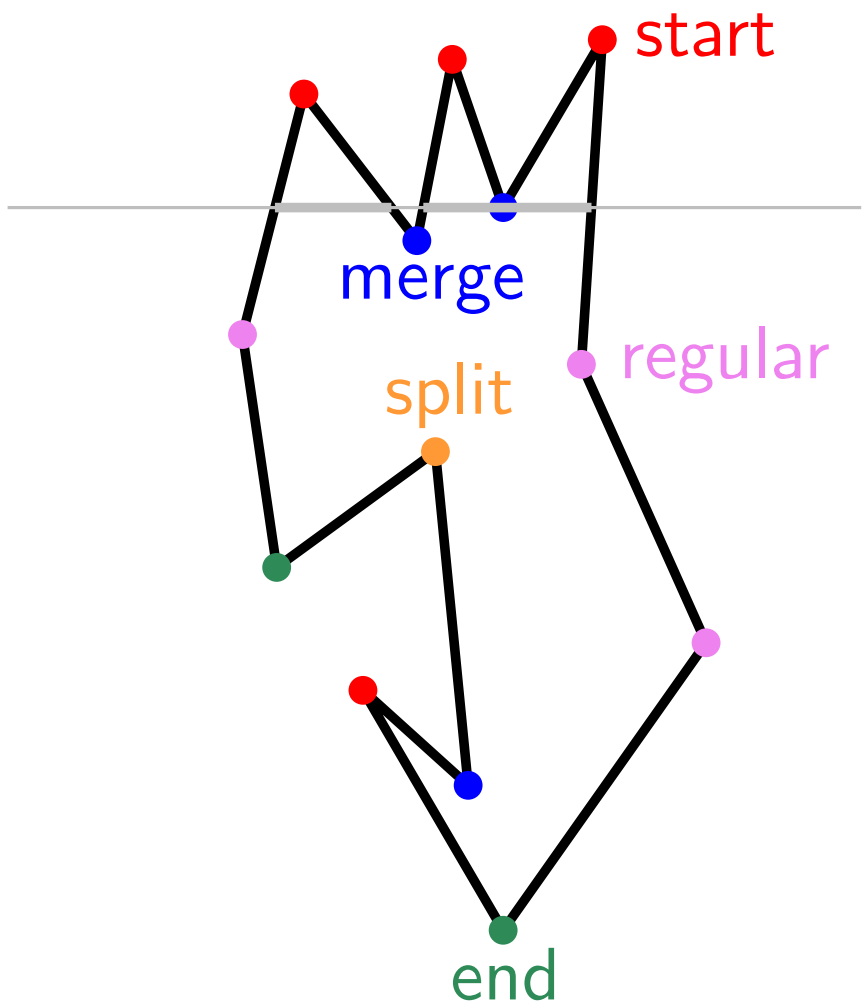

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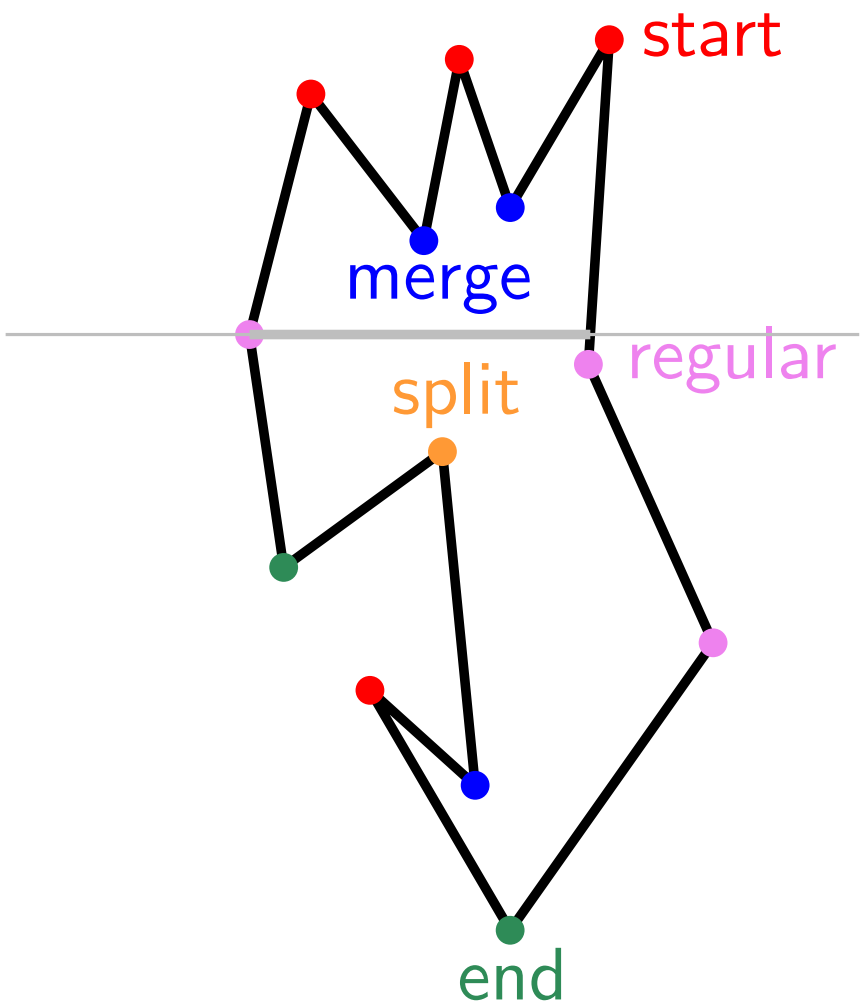

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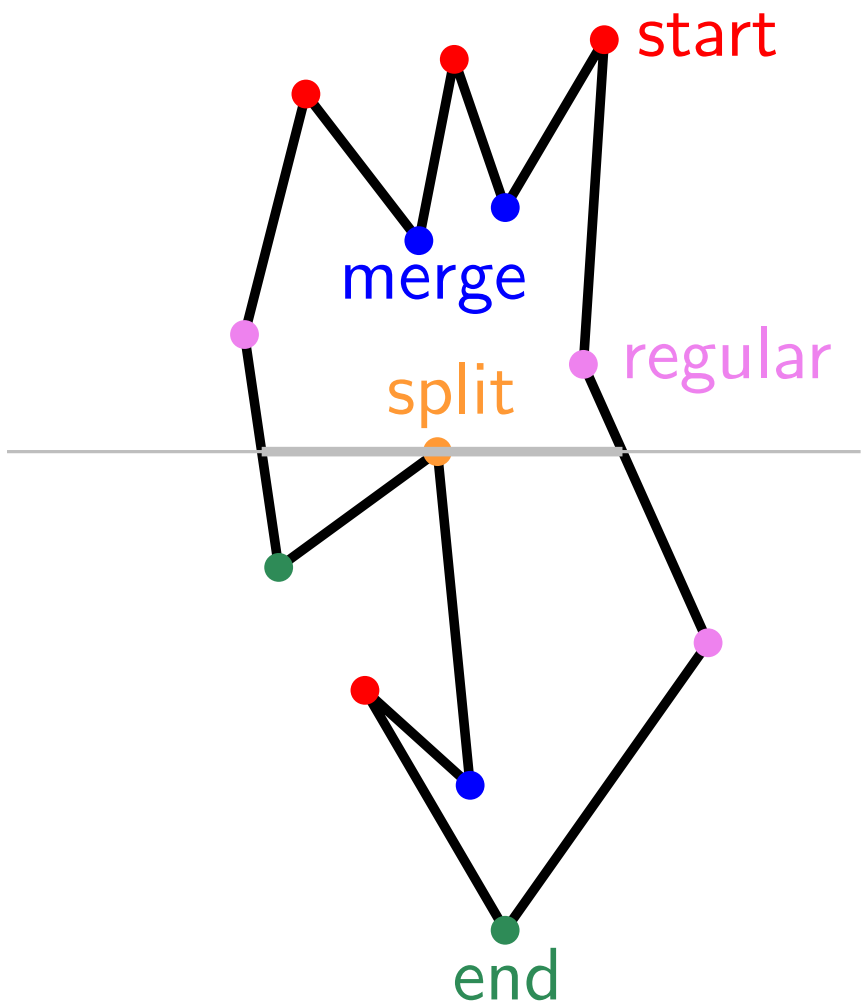

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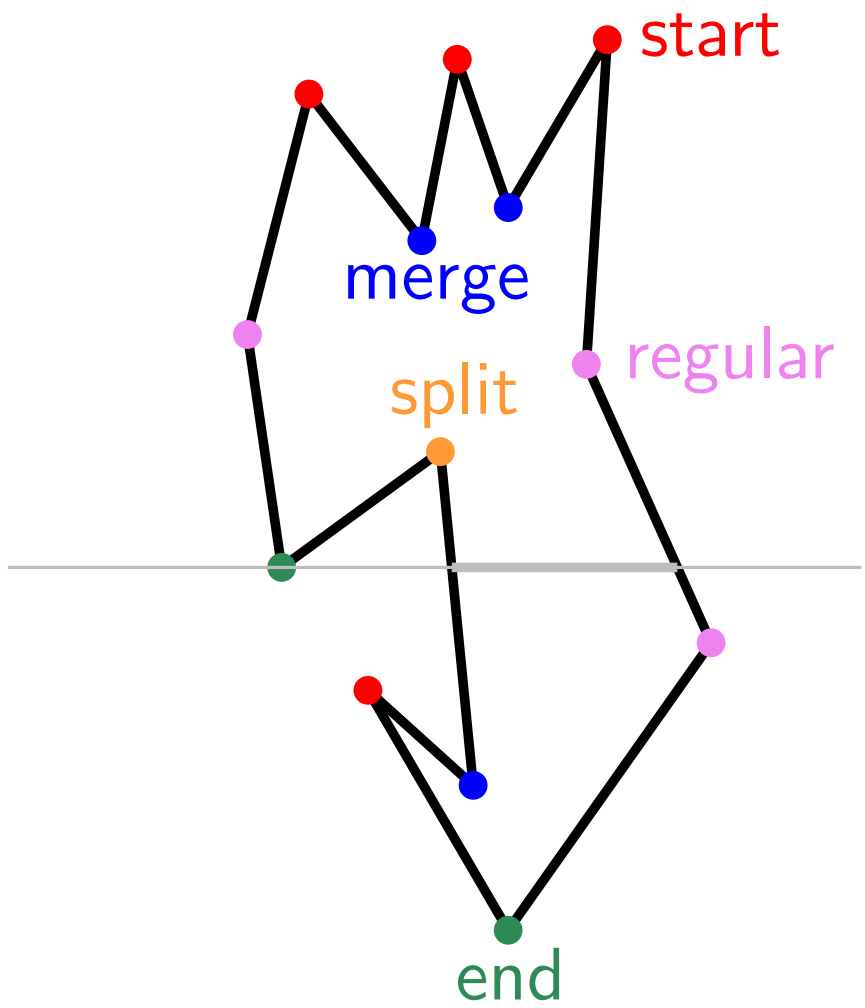
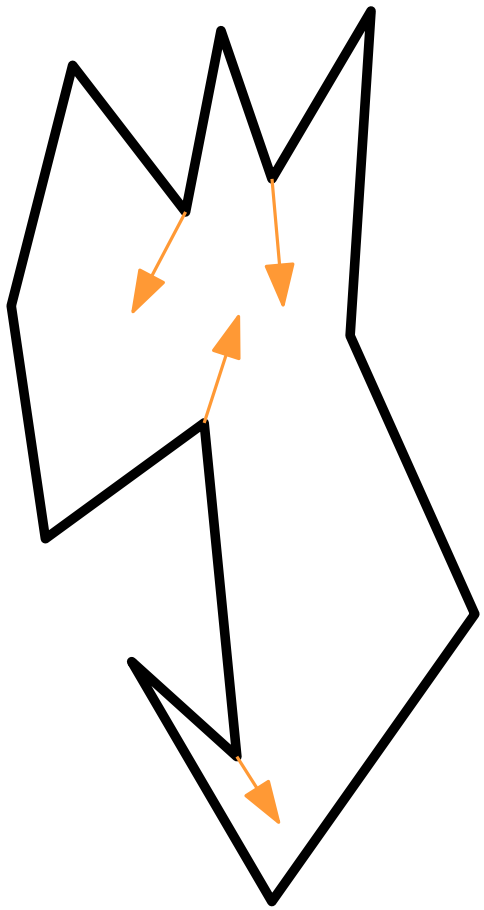


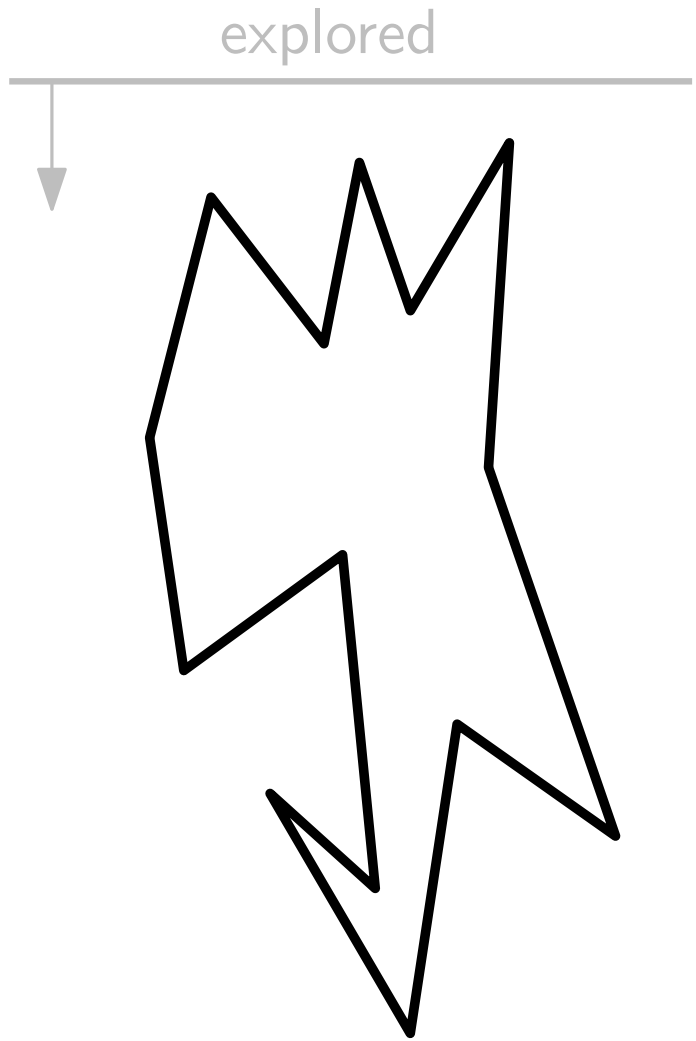
Idea

Idea: "Shoot" a diagonal from every split and merge vertex.
No split and merge $\Rightarrow$ monotone polygon!



Idea

Use horizontal sweep line going down.
Fix split vertices first.
Then go up and fix merge vertices.



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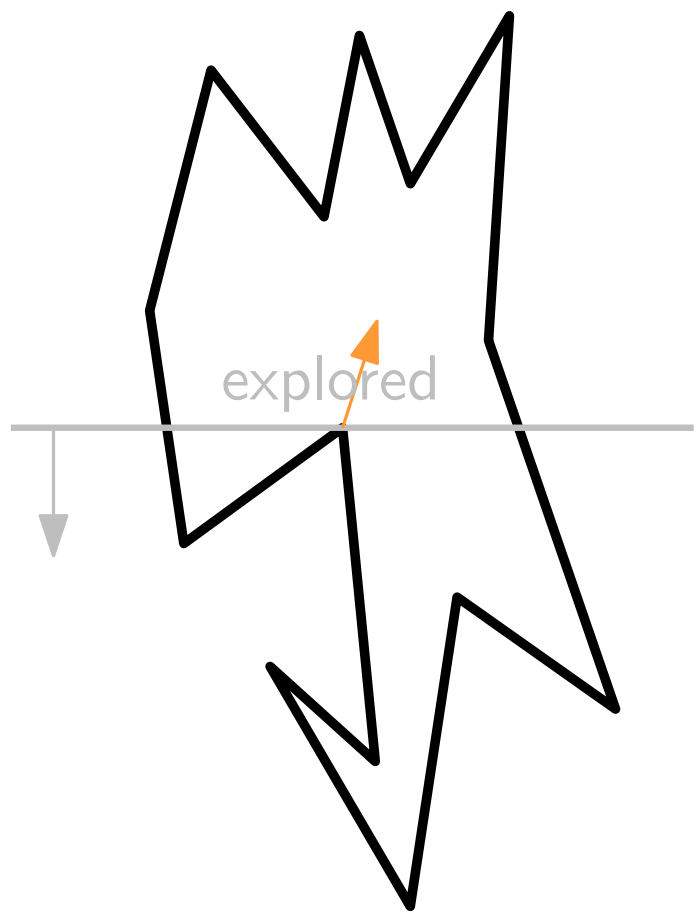

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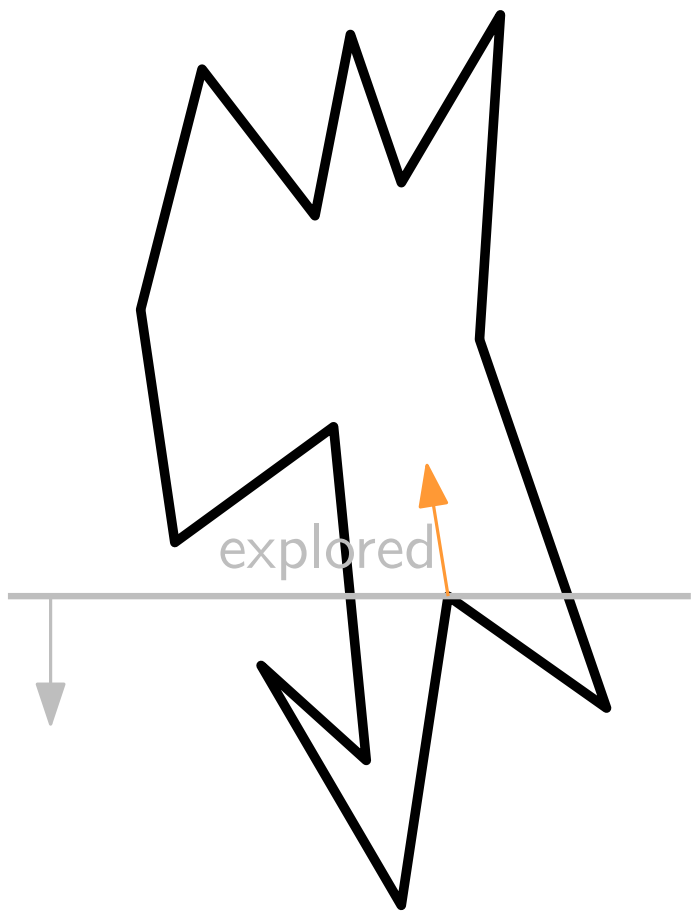
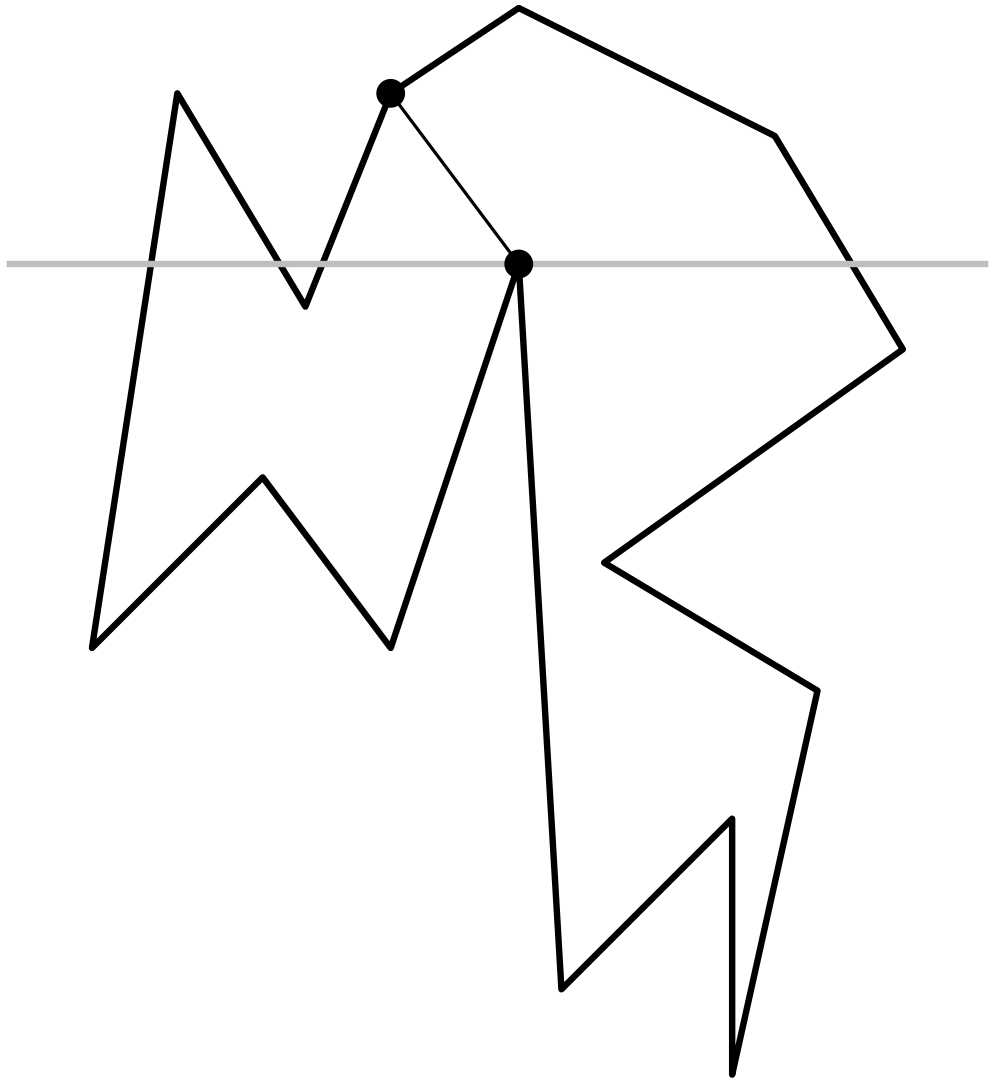


First idea

Make segment to top vertex of line segment to the left?



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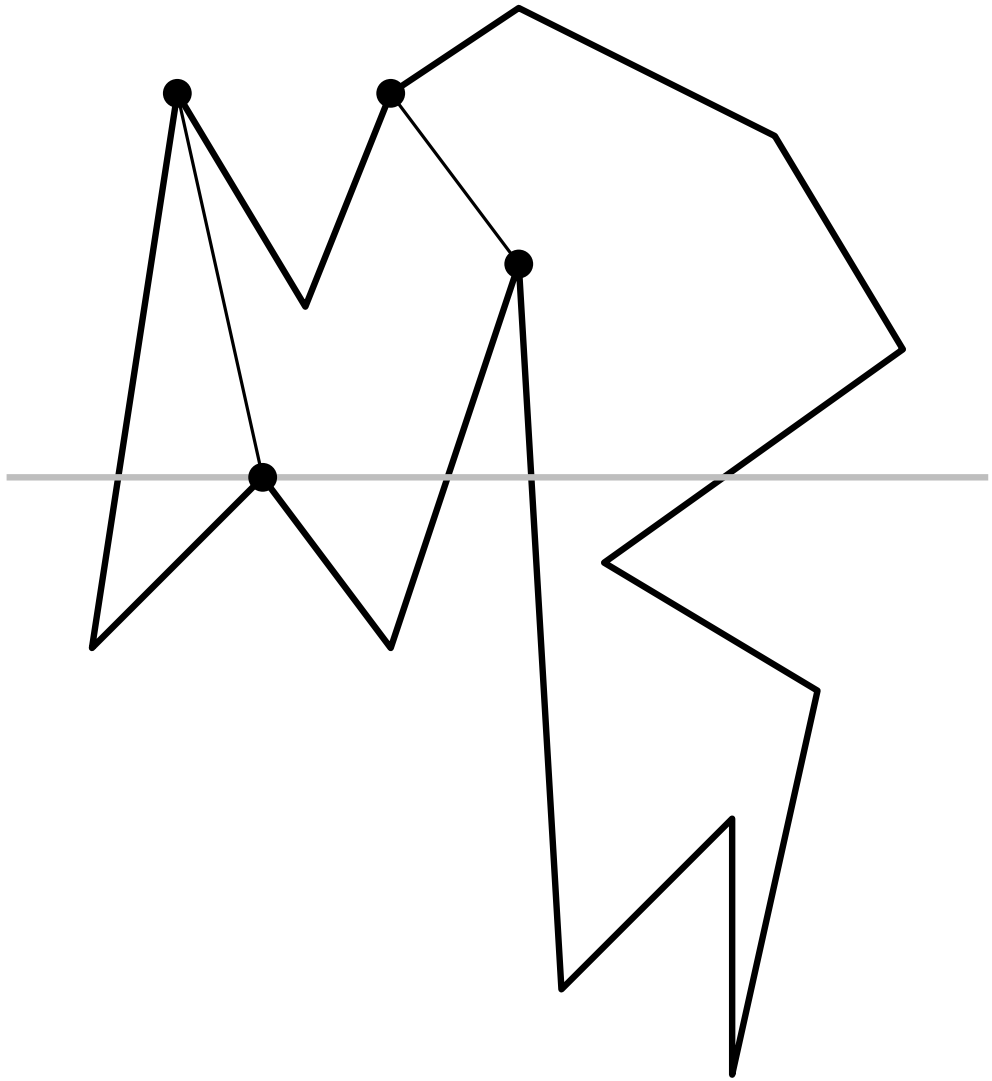

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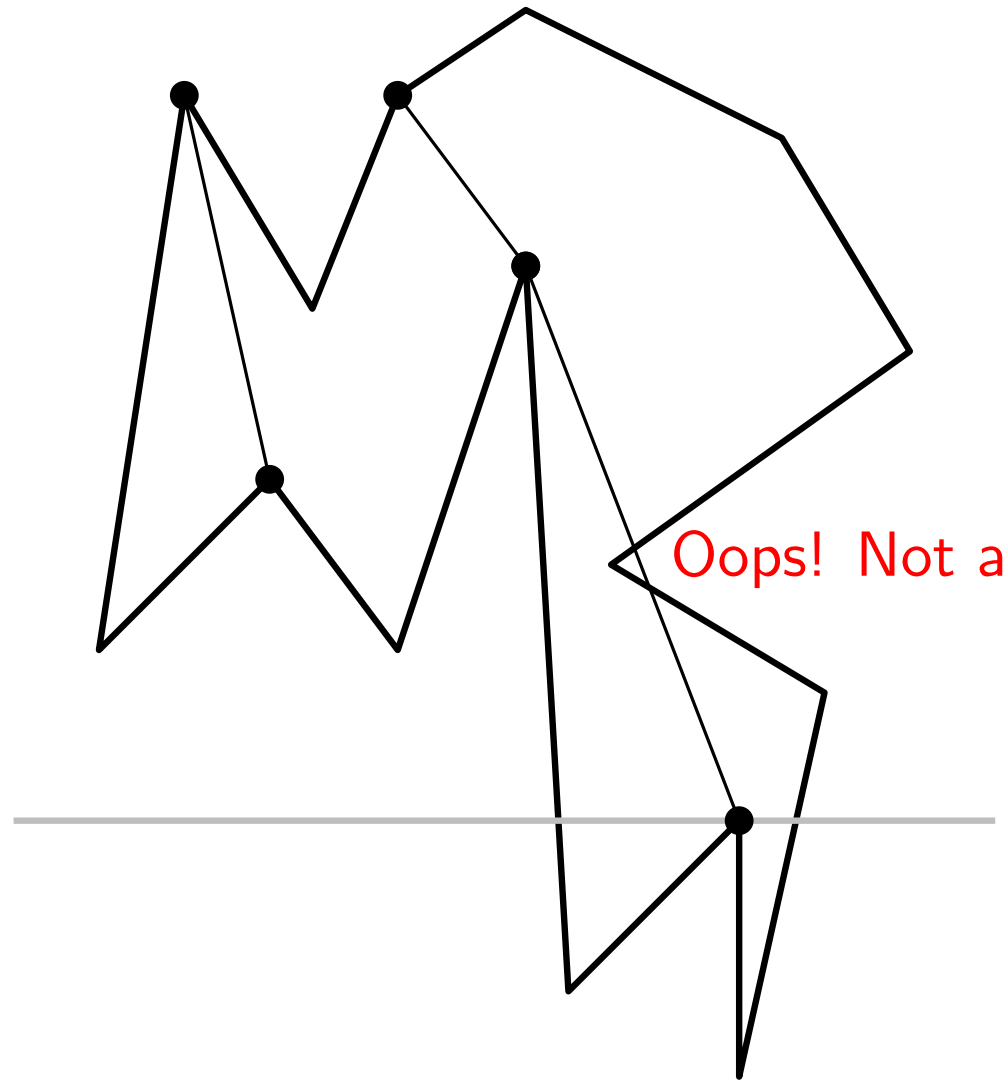
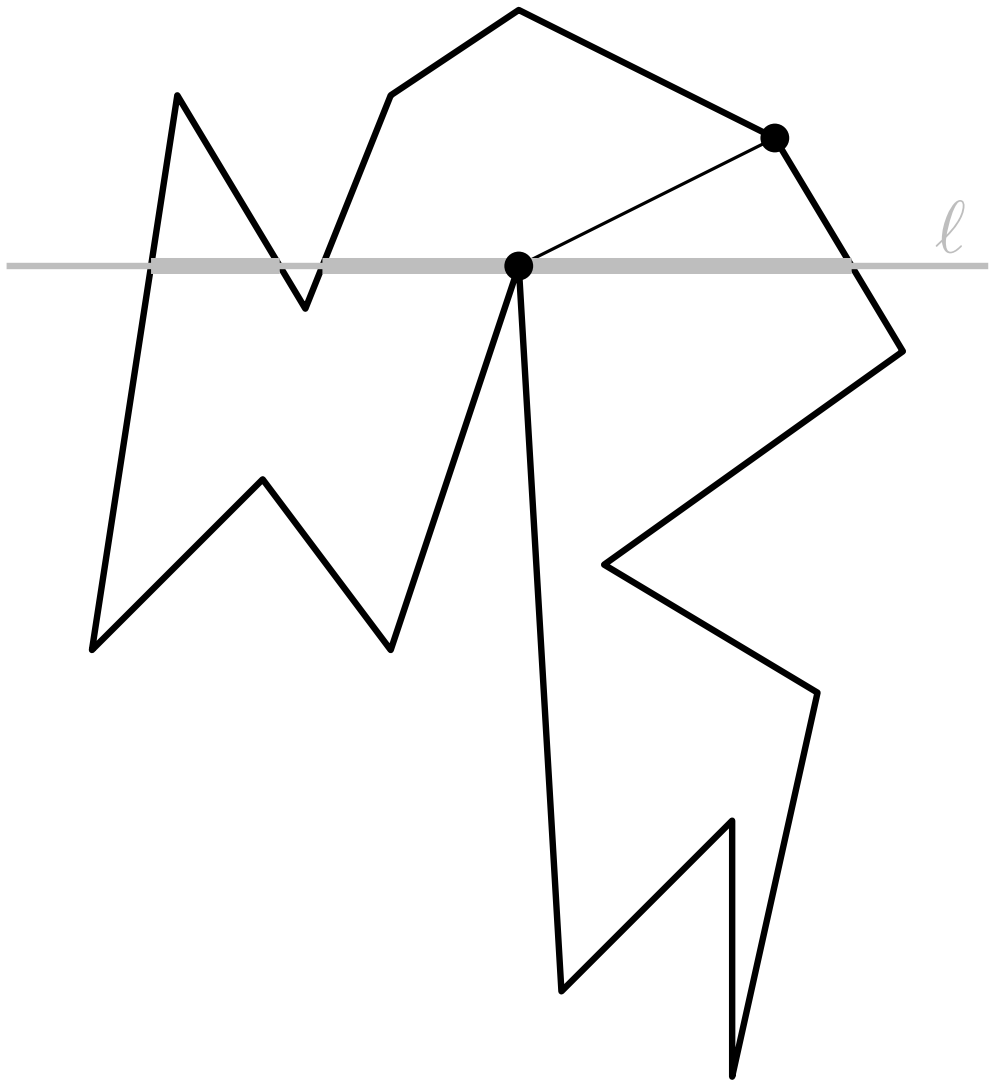


Refined idea

Make segment to previous vertex visited by the component of $\ell \cap P$.



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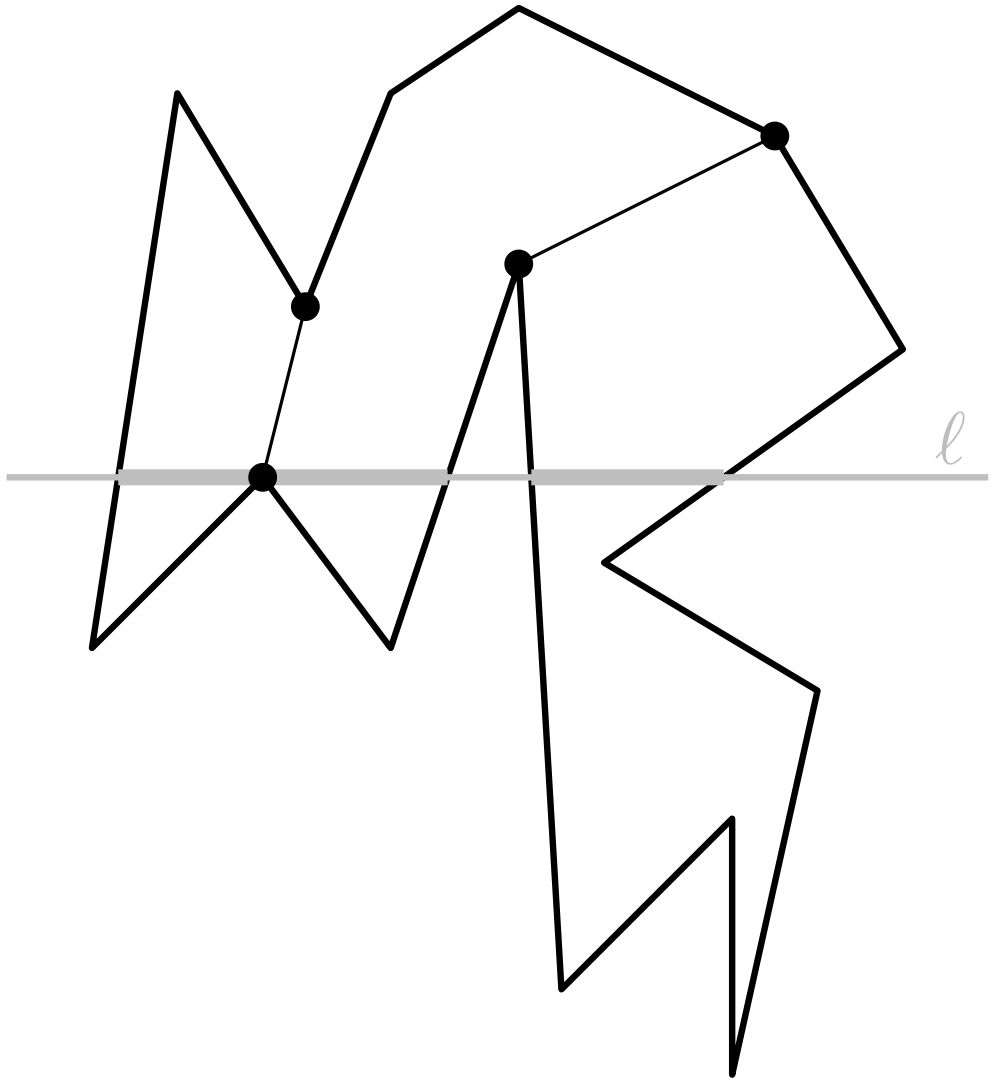

Refined idea

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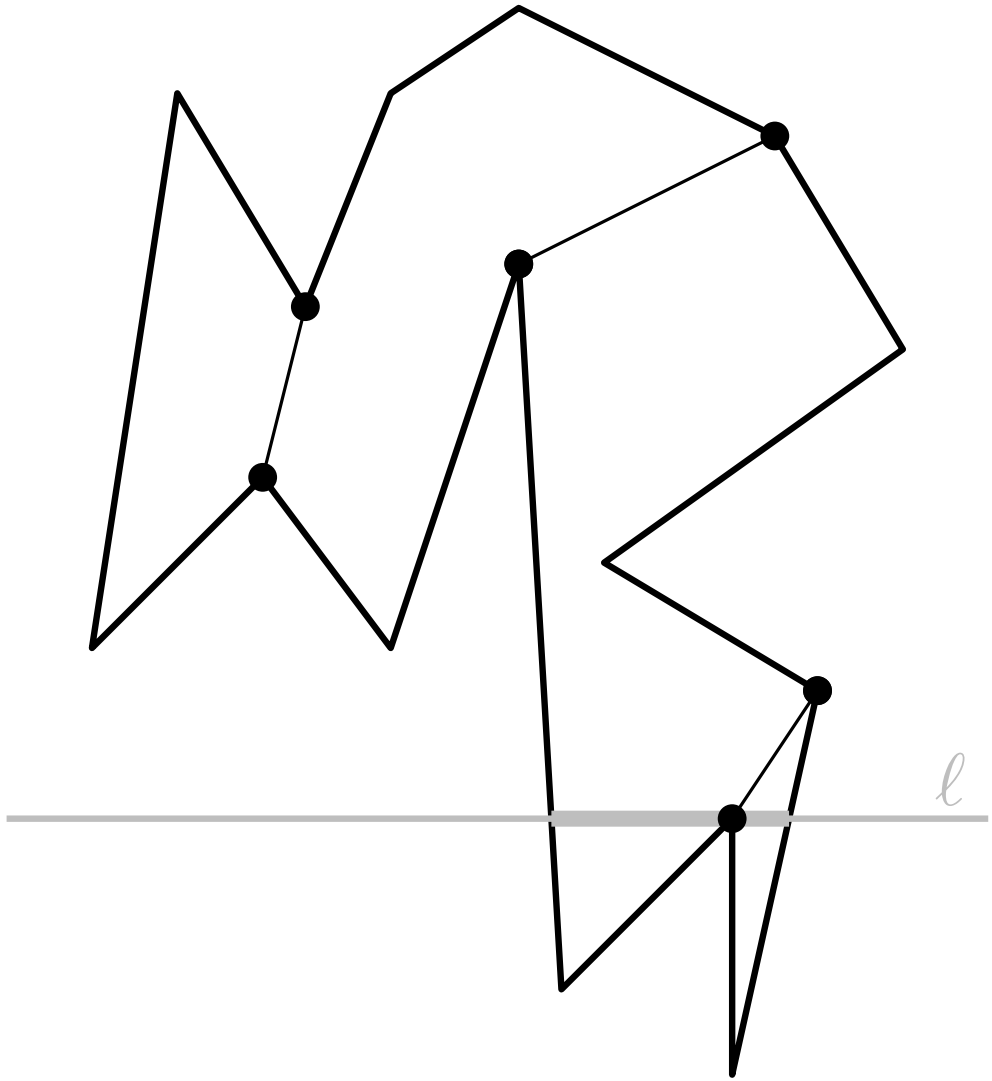
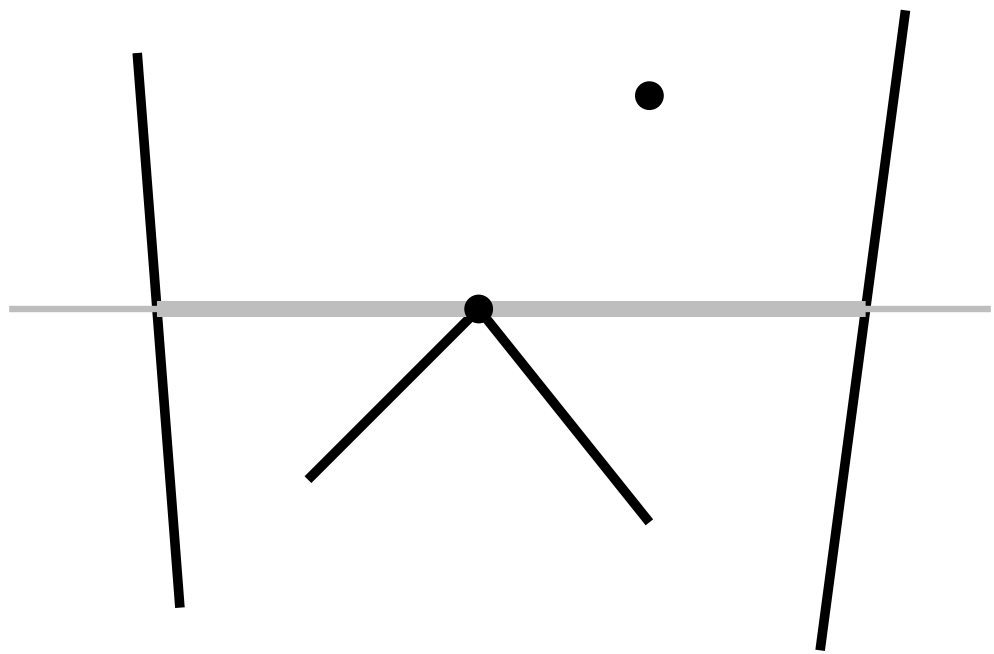
Always a diagonal?
Non-intersecting diagonals?



Idea works!



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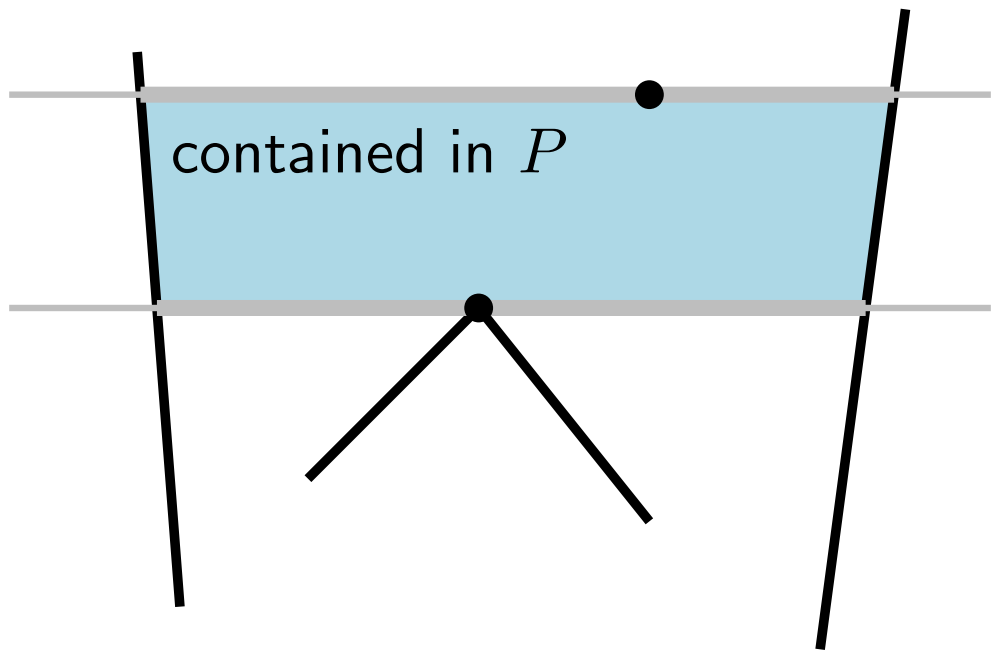

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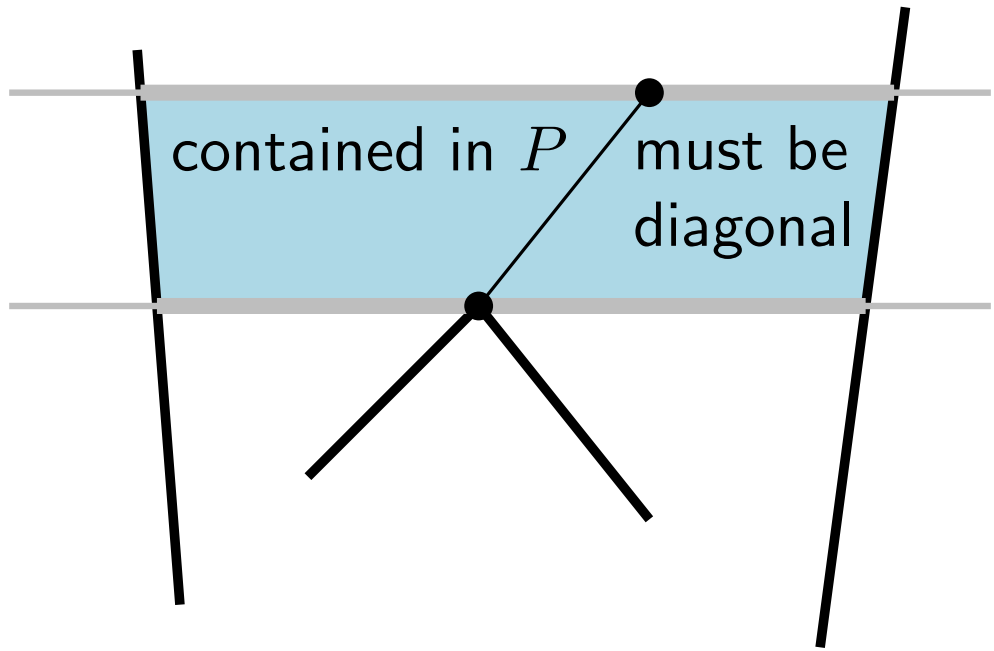
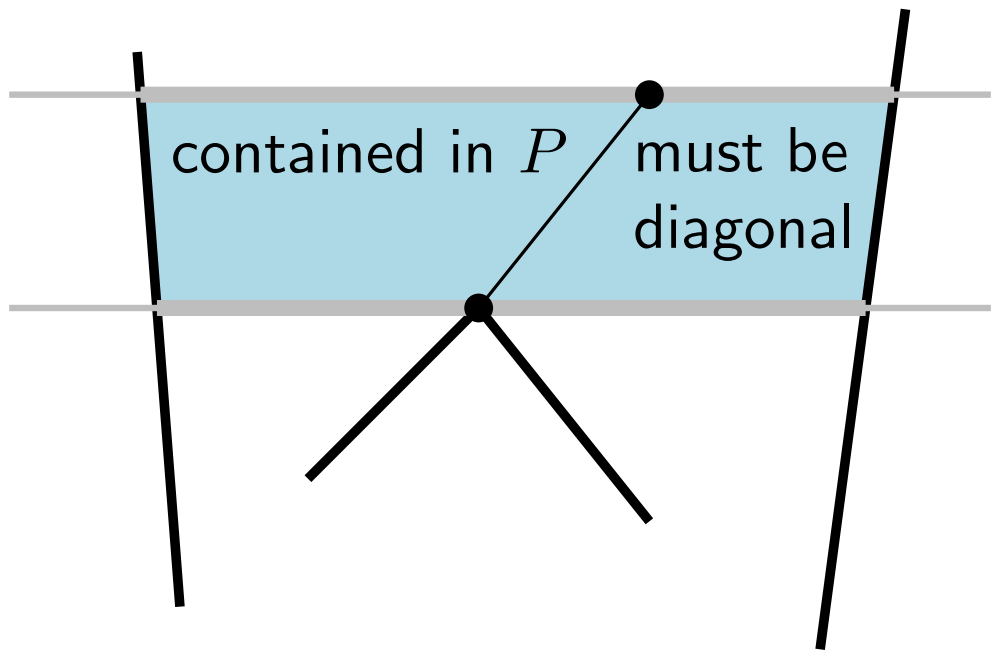


Idea works!

Can new diagonal intersect a previously added?



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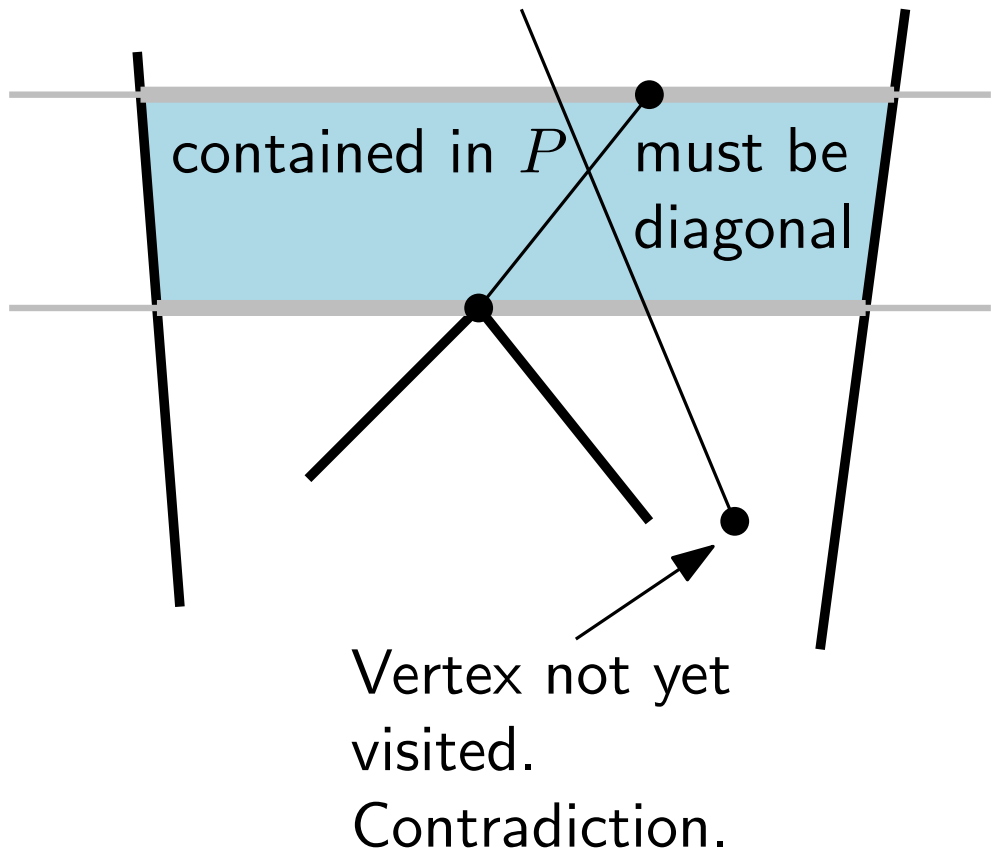

What diagonals are introduced after sweeping down?

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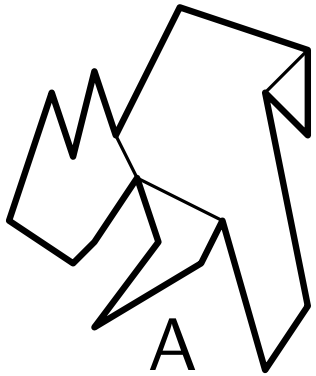
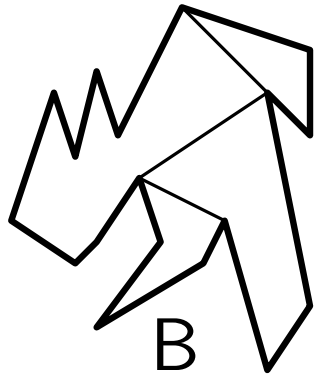
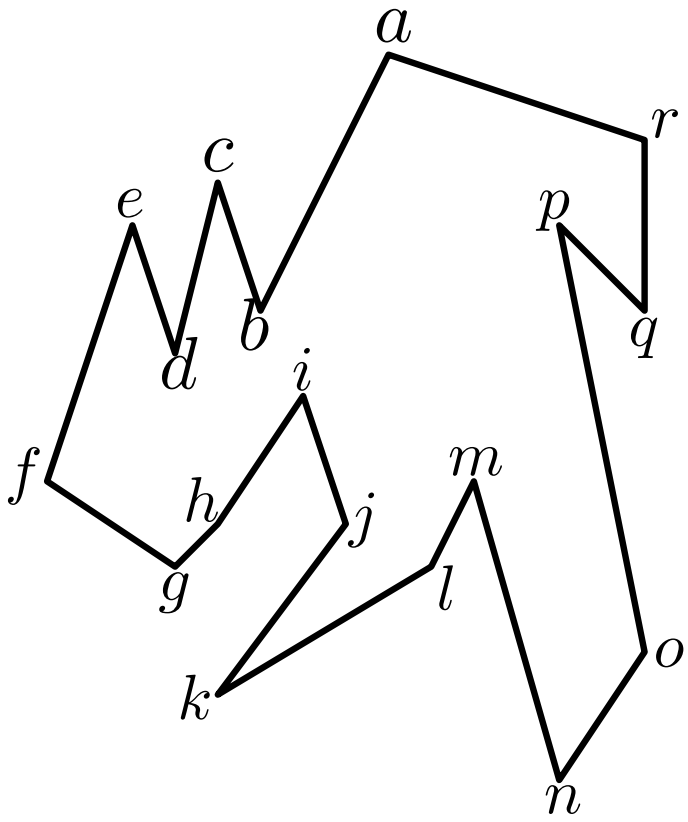
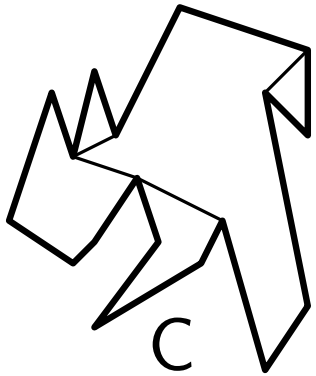
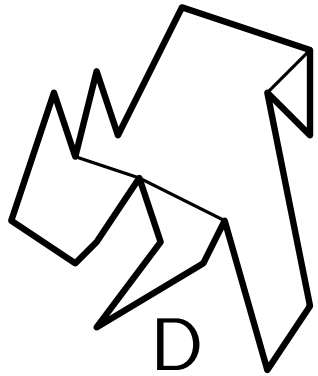
Room name: ABRAHAMSEN3464

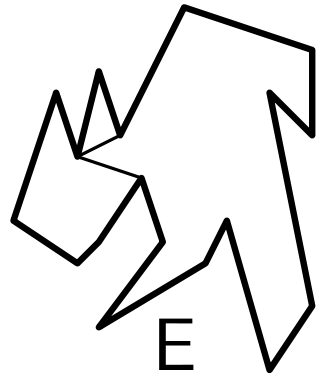








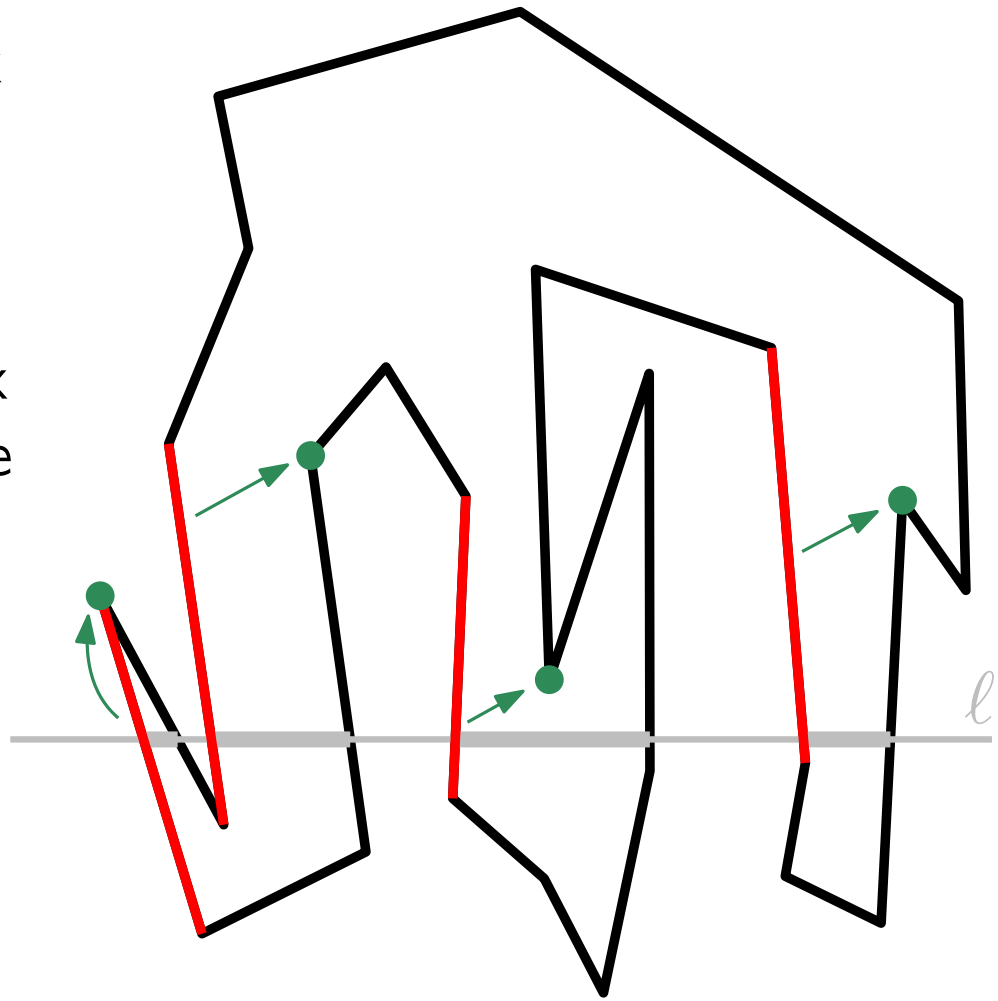




Helpers

Helper of edge e intersected by ℓ with interior of P to the right: Previous vertex visited by this connected component of $\ell \cap P$.

Equivalent: Lowest vertex above ℓ that sees e to the left.

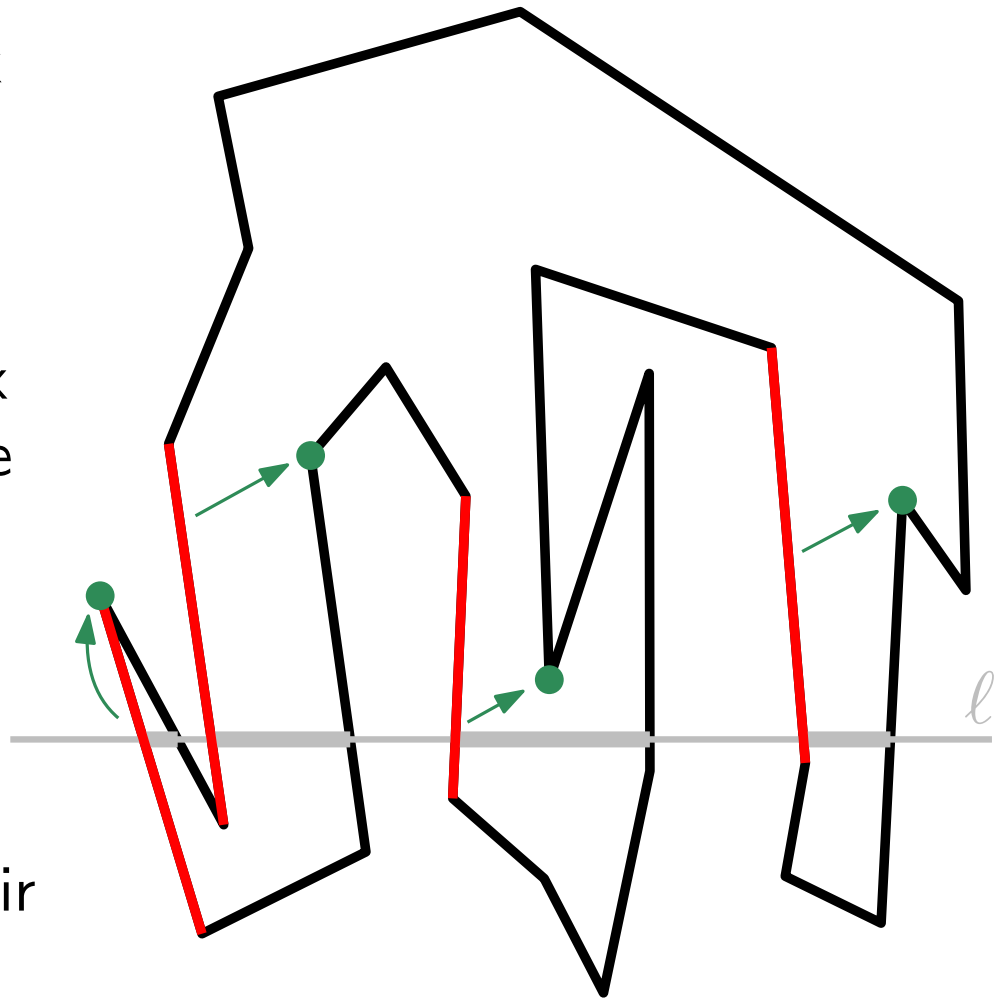


Helpers

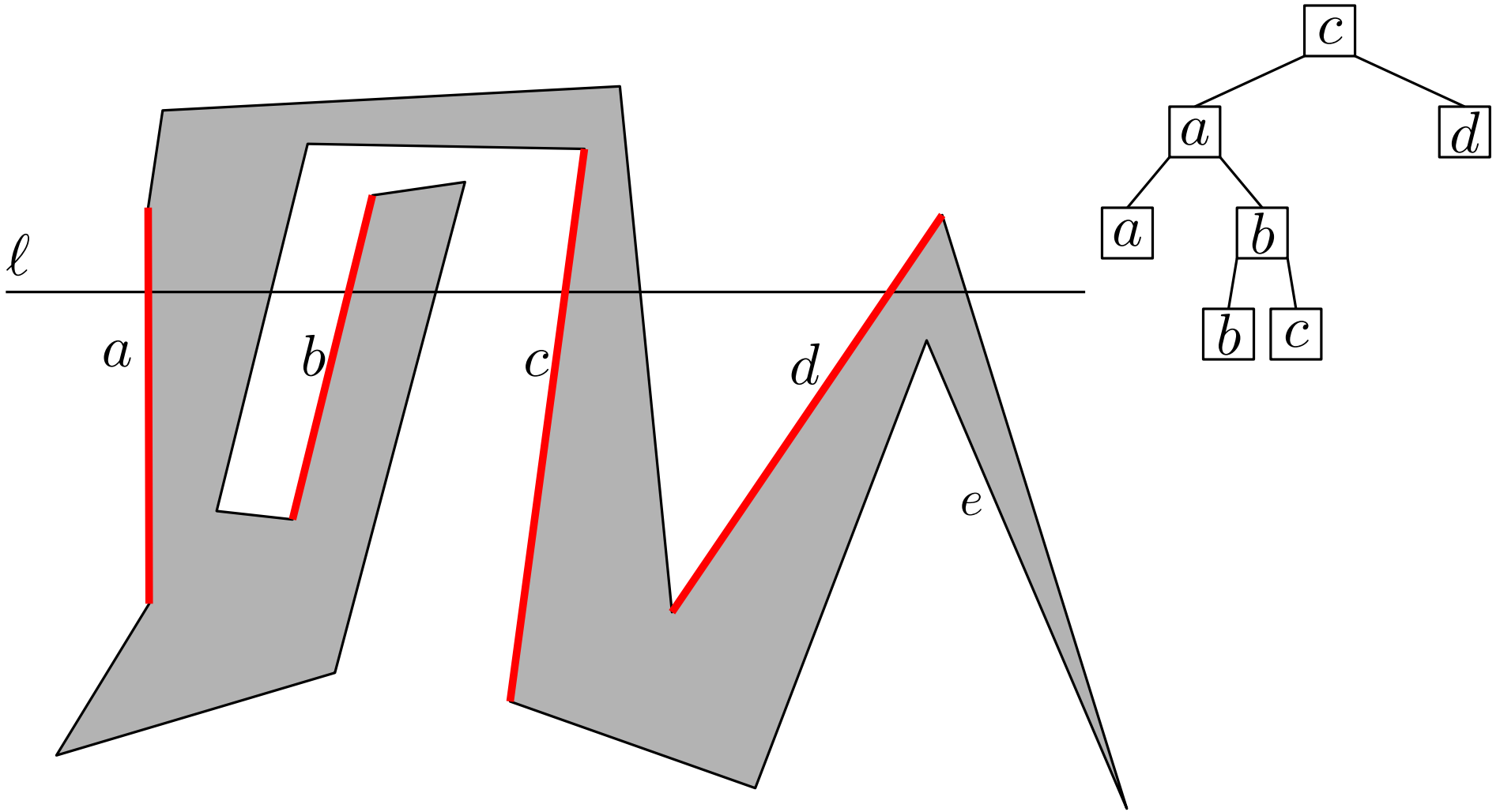
Helper of edge e intersected by ℓ with interior of P to the right: Previous vertex visited by this connected component of $\ell \cap P$.

Equivalent: Lowest vertex above ℓ that sees e to the left.

Intersected edges and their helpers are stored in the *status* structure T.



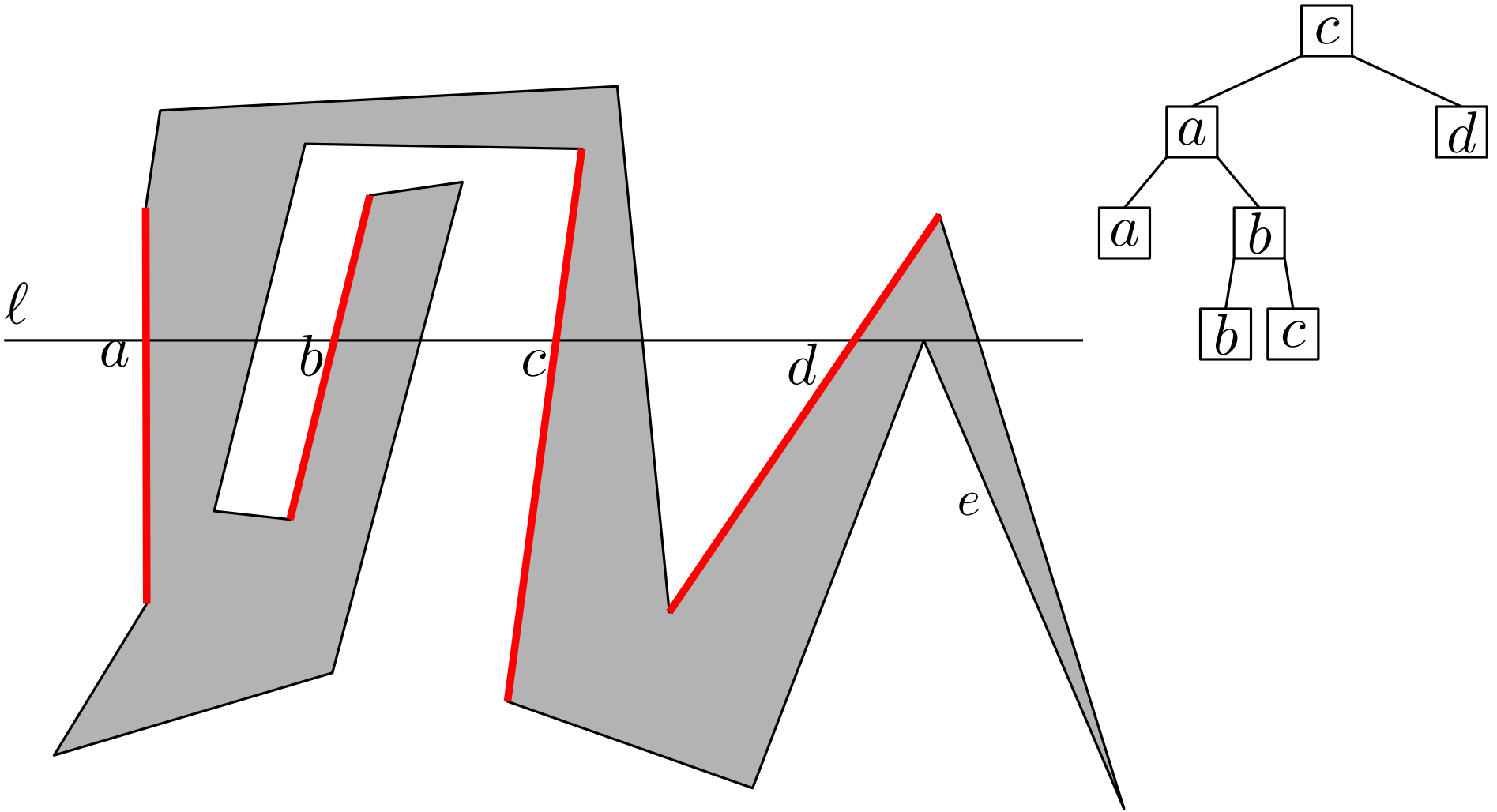
Status



Implement status with balanced binary search tree T.

Sorting order: intersection points with ℓ from left to right.

Status



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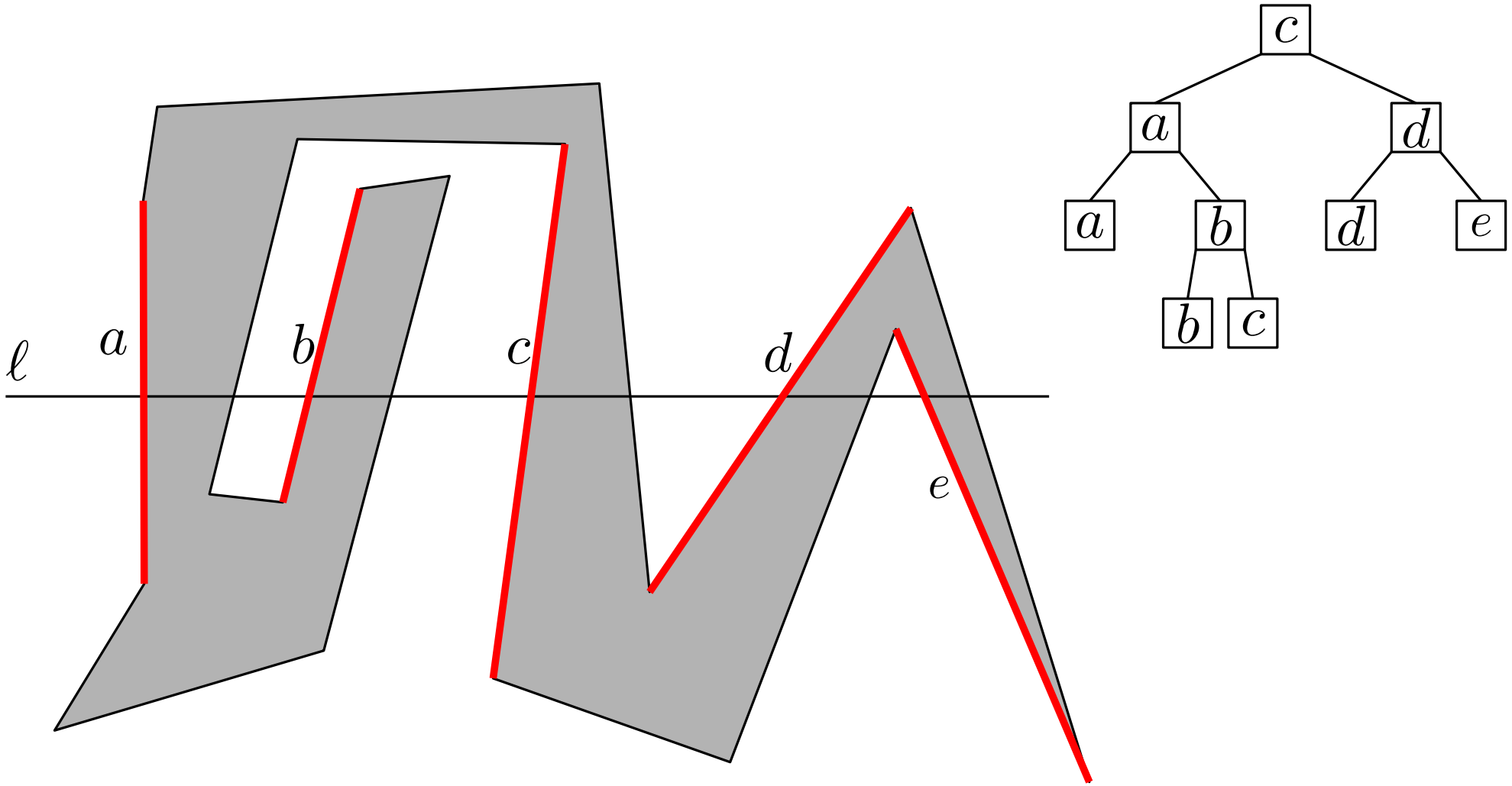

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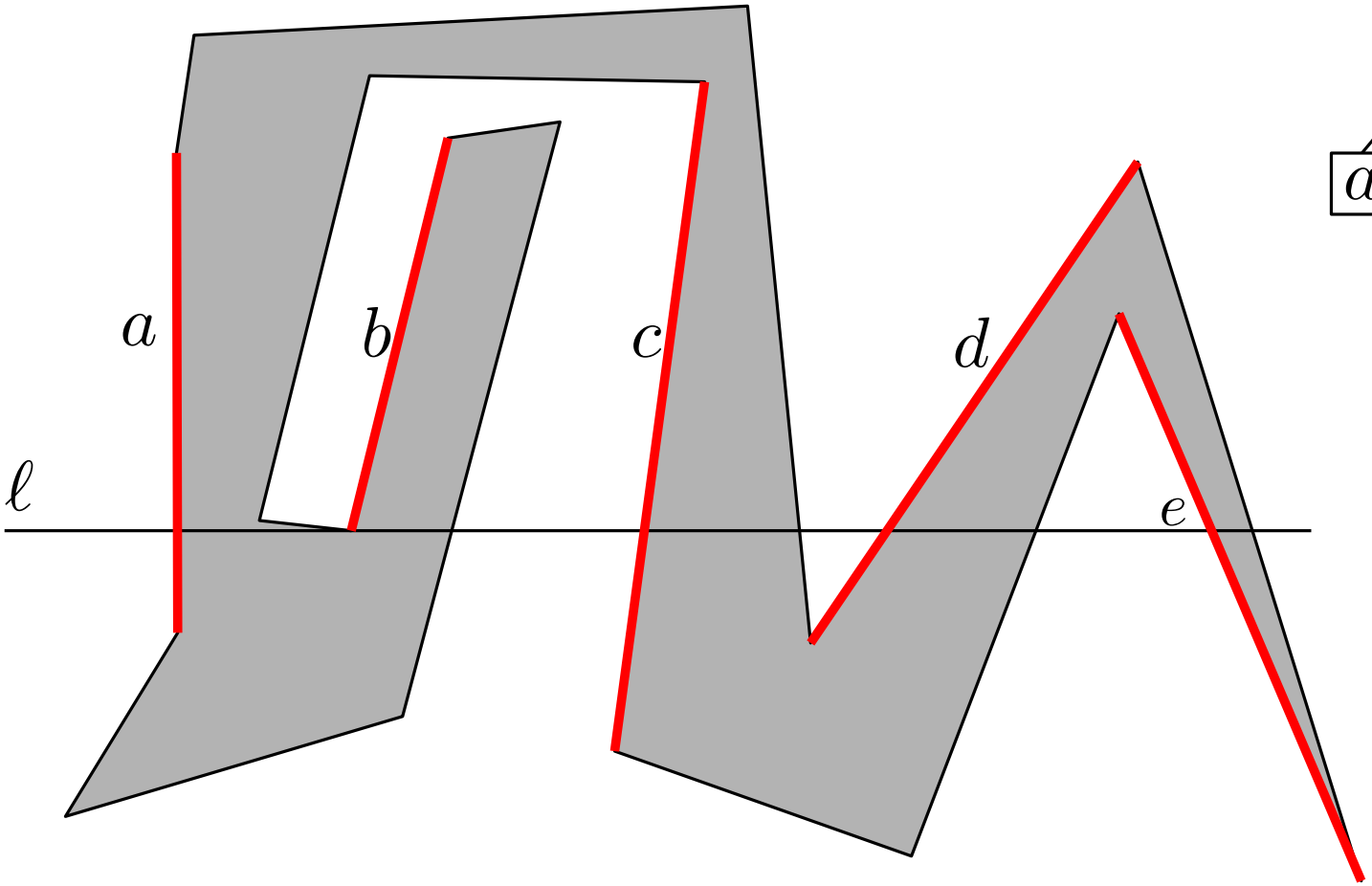
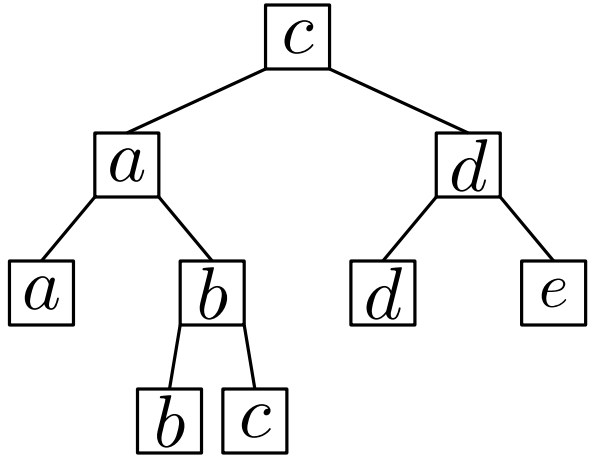
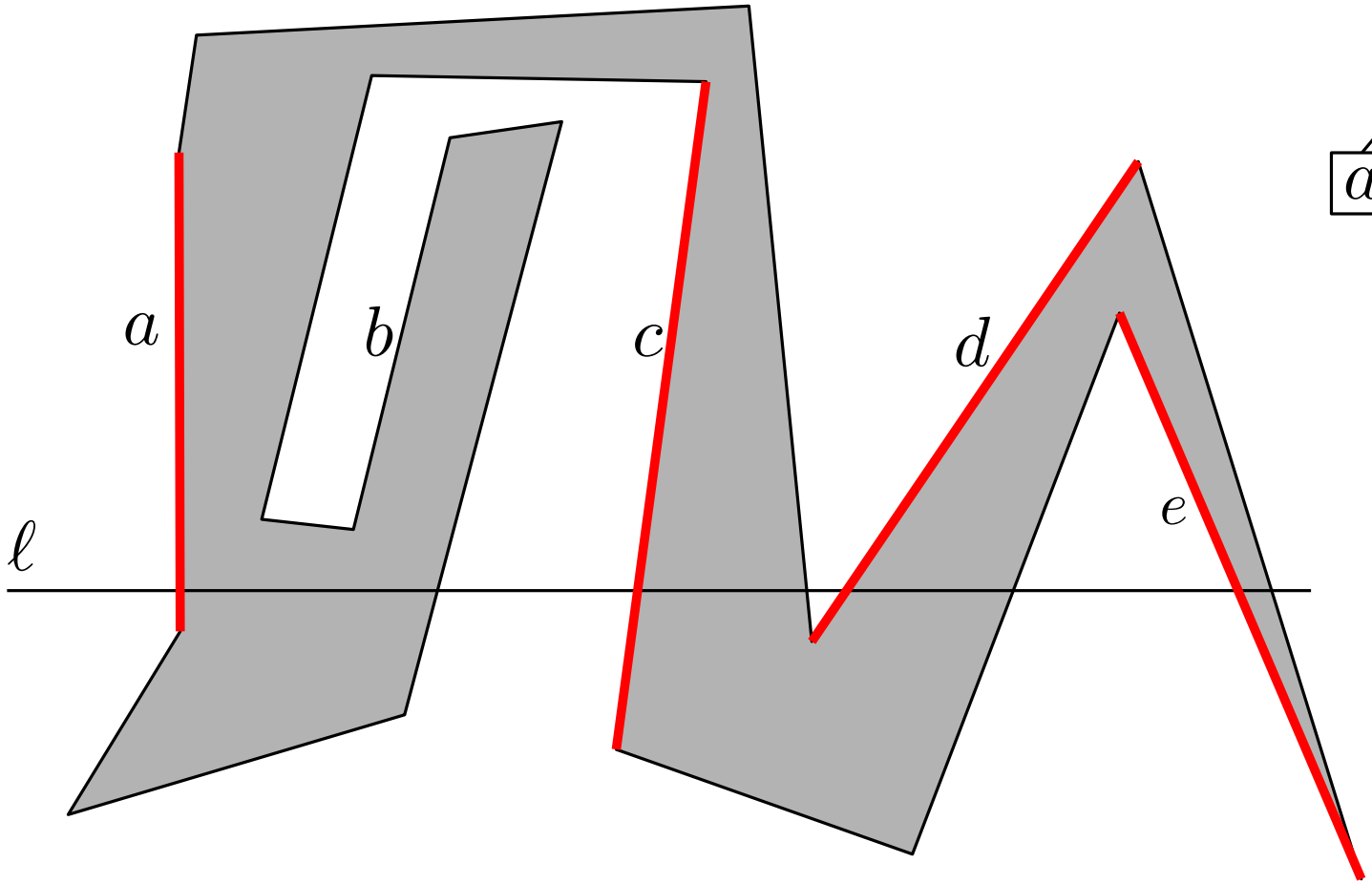
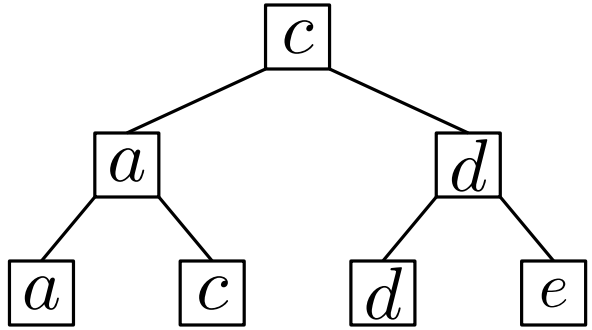




Implement status with balanced binary search tree T.
Sorting order: intersection points with ℓ from left to right.

Status





Implement status with balanced binary search tree T.
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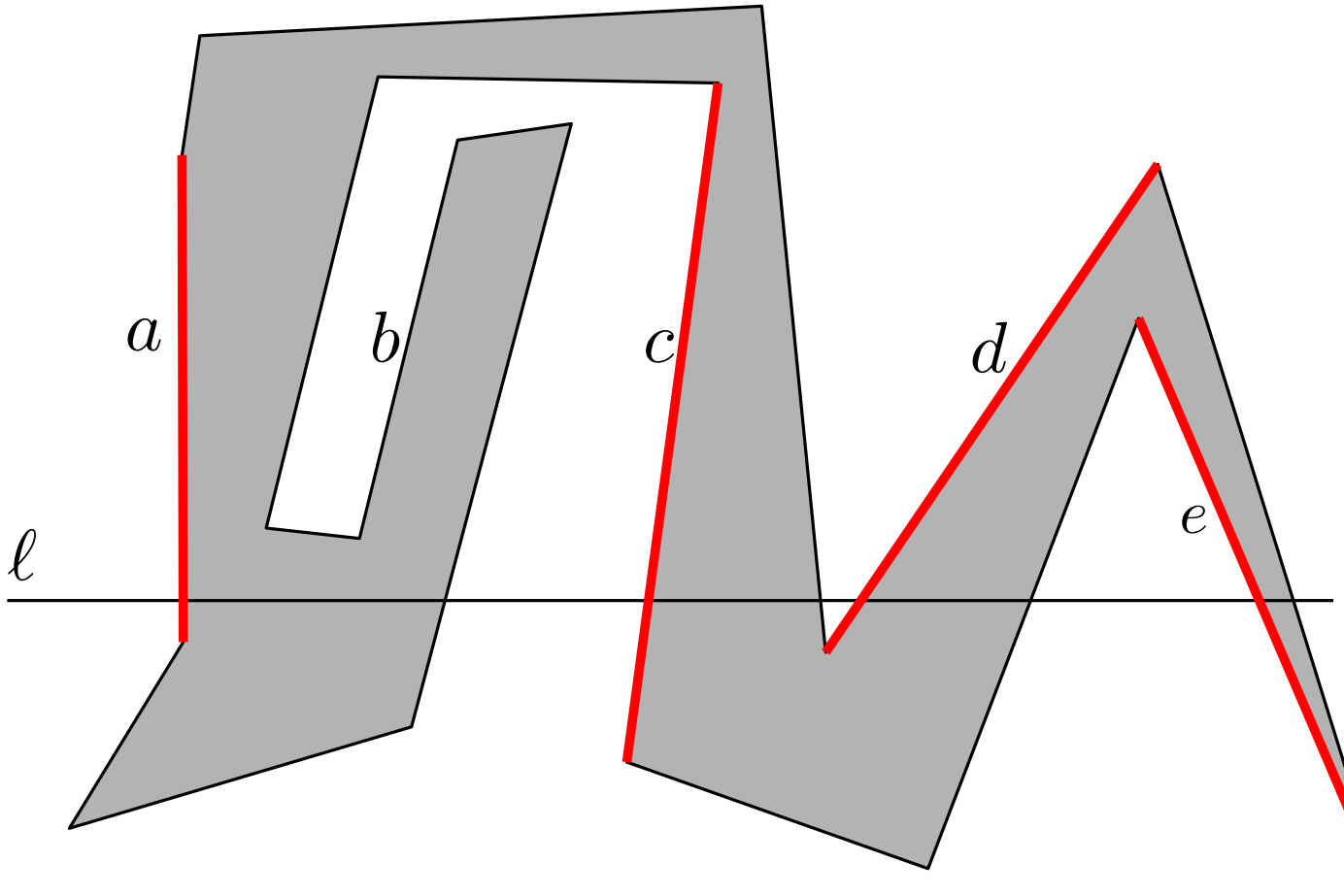
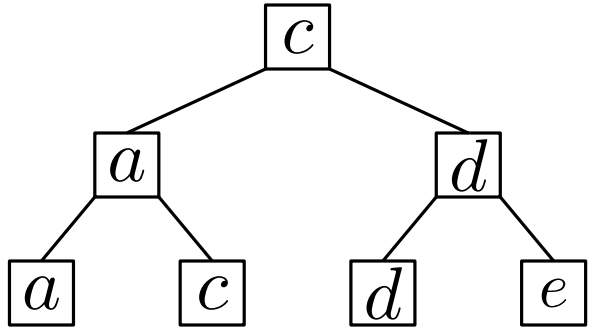


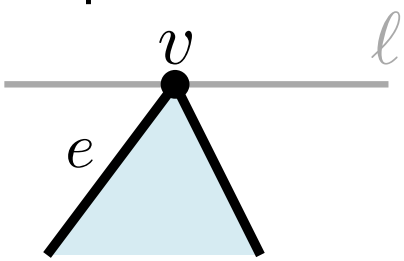


When reaching a new vertex, use $O(\log n)$ time to find the edge in T immediately to the left.

Implement status with balanced binary search tree T.
Sorting order: intersection points with ℓ from left to right.

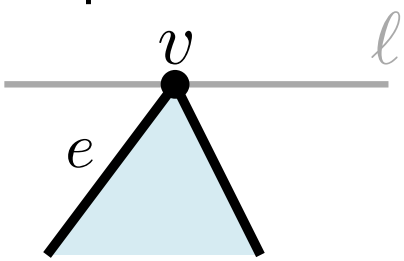
Events

Start vertex: Insert e in T with helper v.

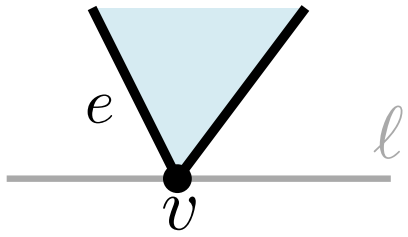


Events

Start vertex: Insert e in T with helper v.

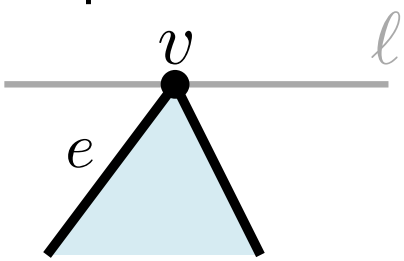


End vertex: Remove e from T.

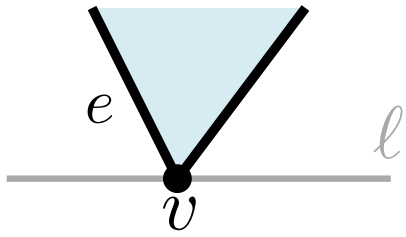


Events

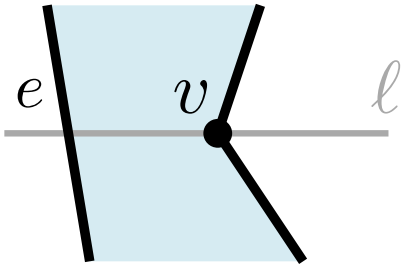
Start vertex: Insert e in T with helper v.



End vertex: Remove e from T.

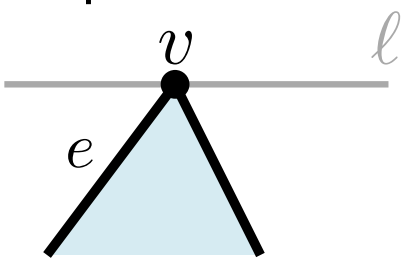


Regular vertex with P to the left: Update helper of e to v.

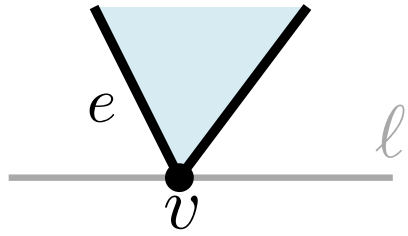


Events

Start vertex: Insert e in T with helper v.



End vertex: Remove e from T.



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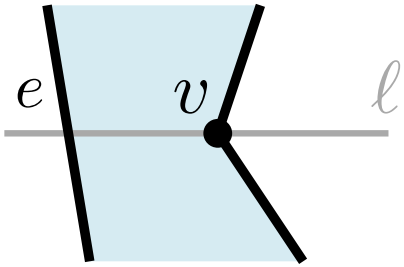
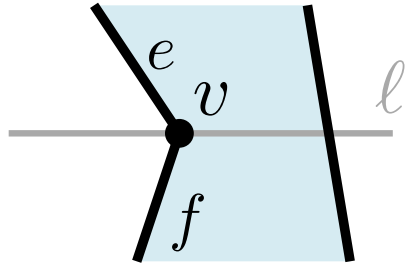
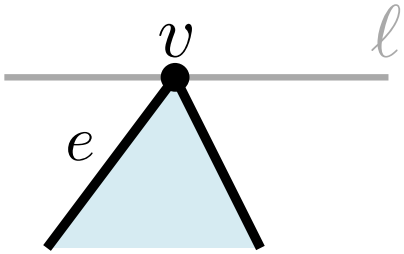


Regular vertex with P to the right: Replace e by f in T with helper v.

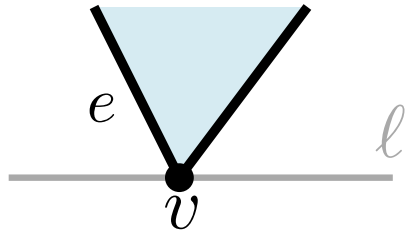


Events

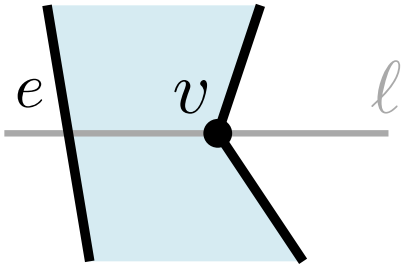
Start vertex: Insert e in T with helper v.



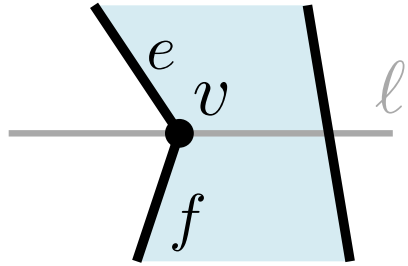
End vertex: Remove e from T.



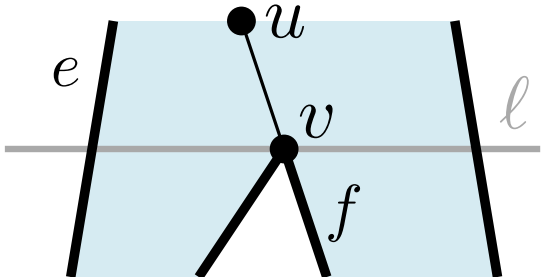
Regular vertex with P to the left: Update helper of e to v.



Regular vertex with P to the right: Replace e by f in T with helper v.



Split vertex: Add diagonal to helper u of e. Update helper of e to v. Add f to T with helper v.



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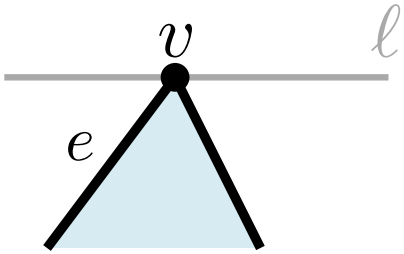
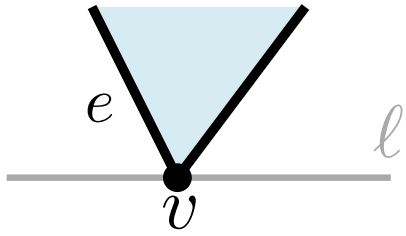
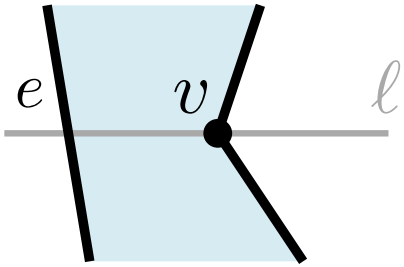


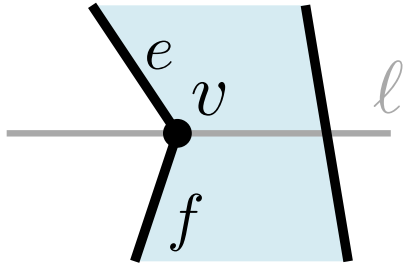
End vertex: Remove e from T.



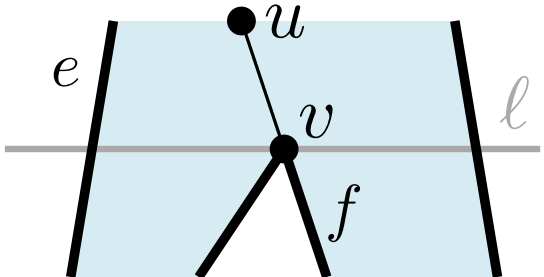
Regular vertex with P to the left: Update helper of e to v.



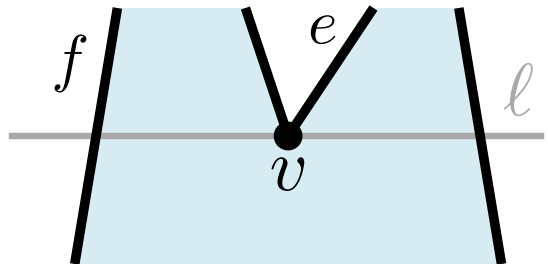
Regular vertex with P to the right: Replace e by f in T with helper v.



Split vertex: Add diagonal to helper u of e. Update helper of e to v. Add f to T with helper v.

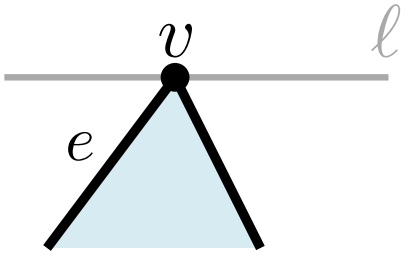


Merge vertex: Remove e from T. Update helper of f to v.

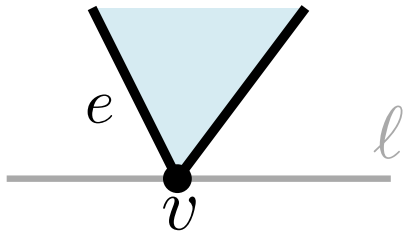


Events

Start vertex: Insert e in T with helper v.

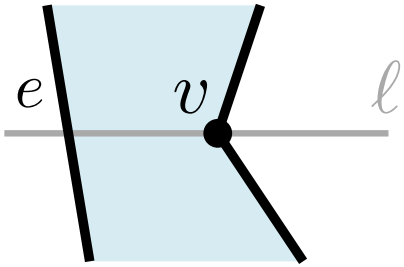


End vertex: Remove e from T.

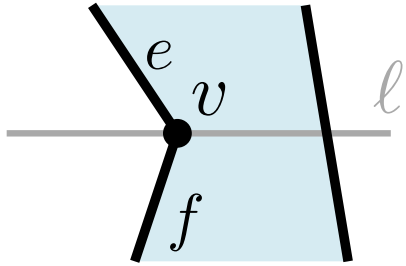


Each event takes time $O(\log n)$.

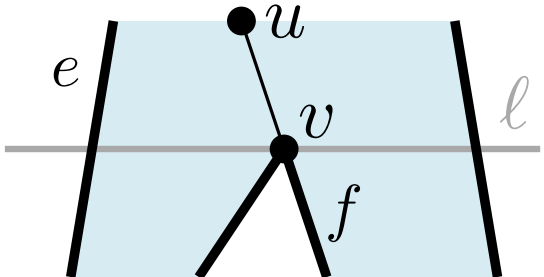
Regular vertex with P to the left: Update helper of e to v.



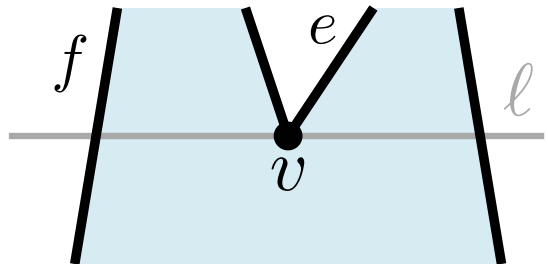
Regular vertex with P to the right: Replace e by f in T with helper v.



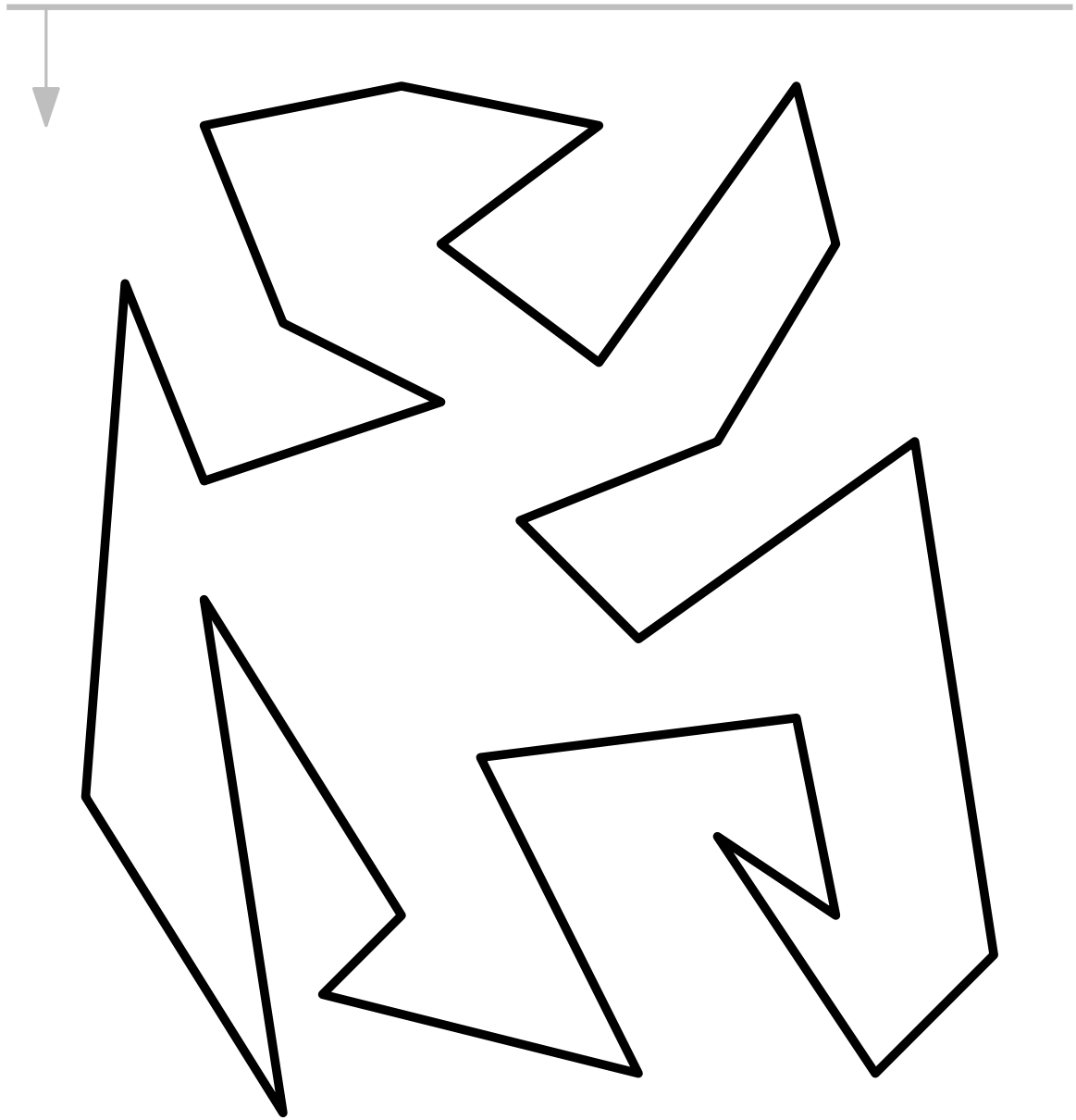
Split vertex: Add diagonal to helper u of e. Update helper of e to v. Add f to T with helper v.



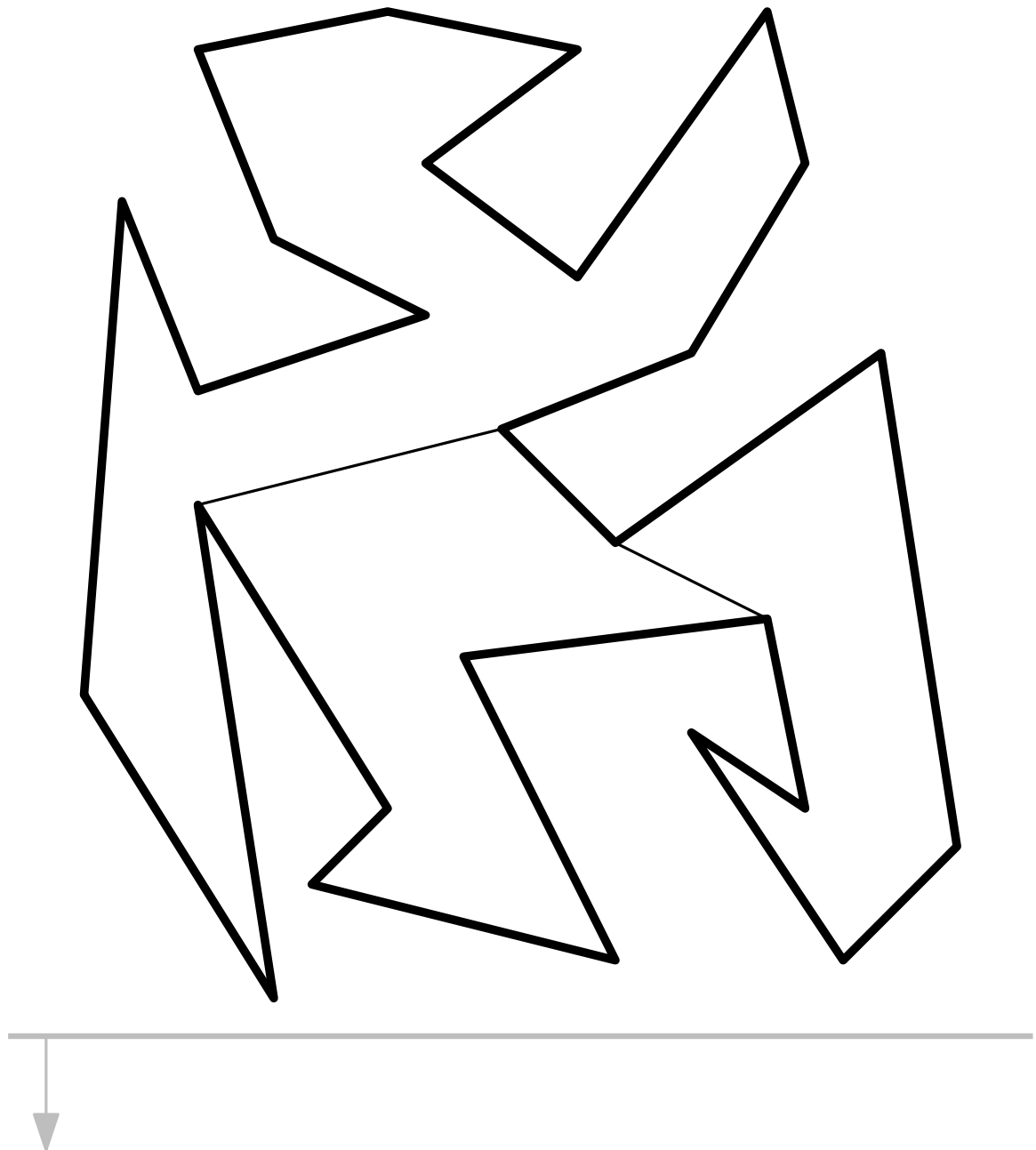
Merge vertex: Remove e from T. Update helper of f to v.



Sweep down, then up

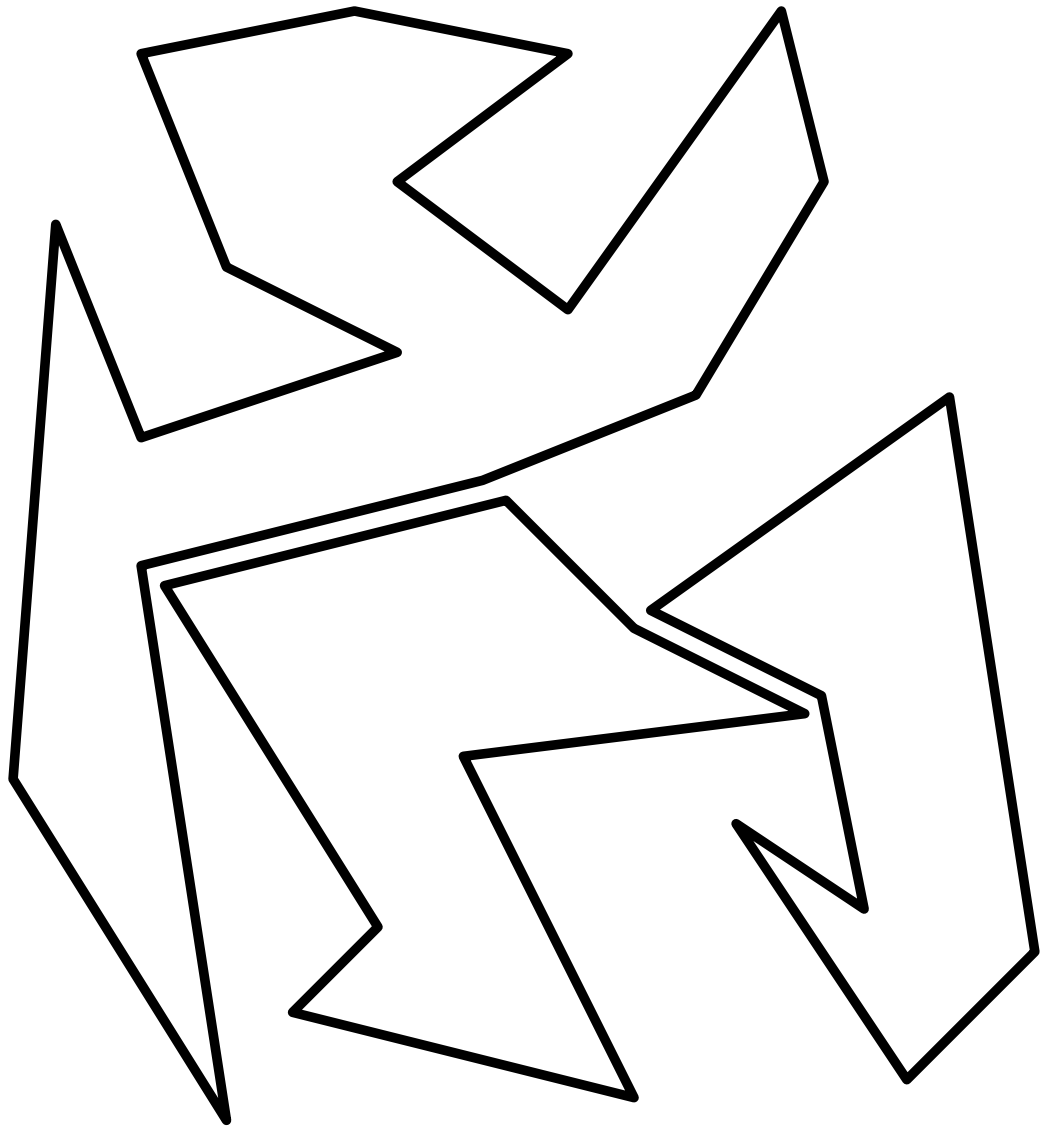


Sweep down, then up



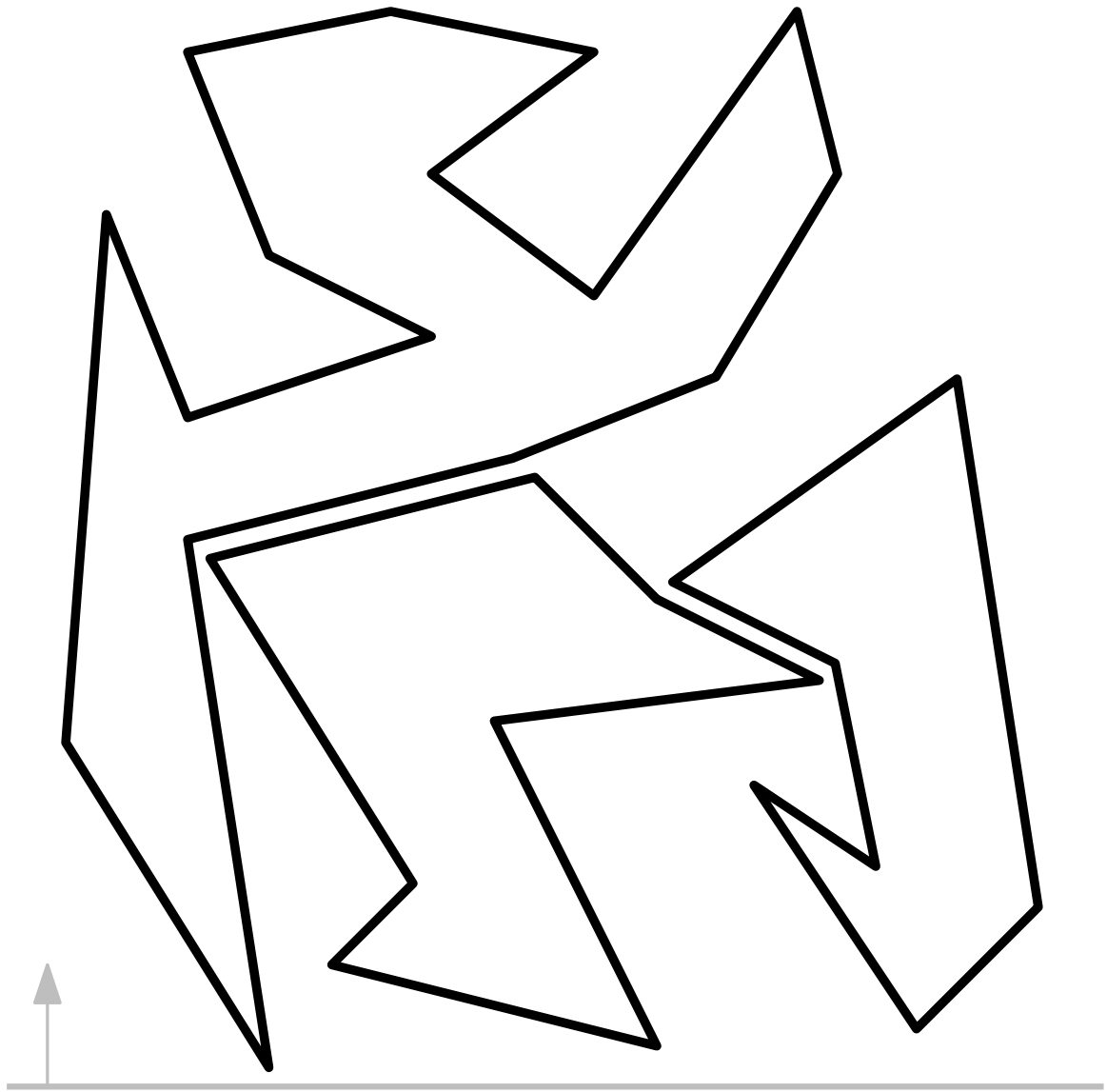
Sweep down, then up

From below:
Each polygon
separately!



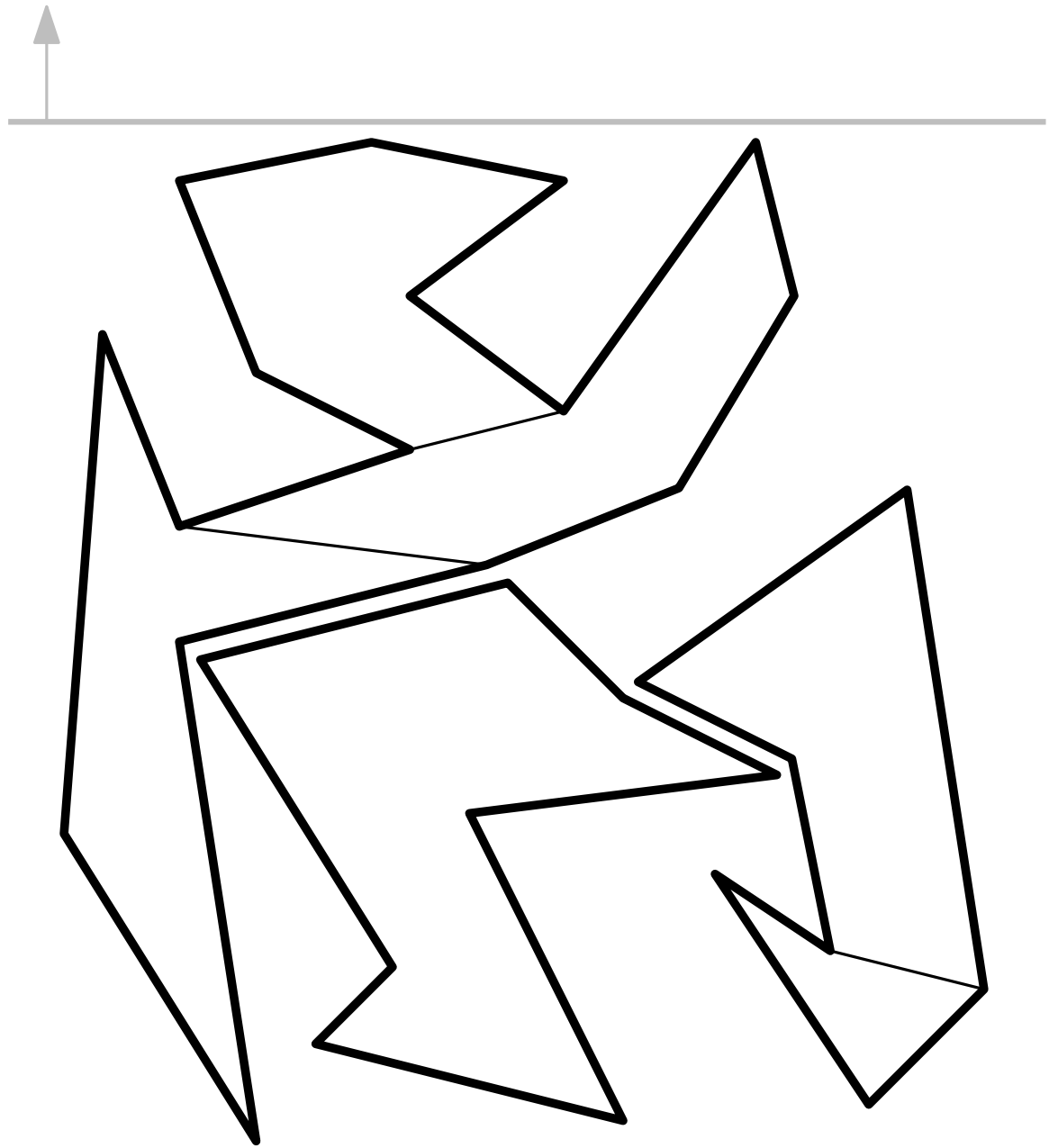
Sweep down, then up

From below:
Each polygon
separately!

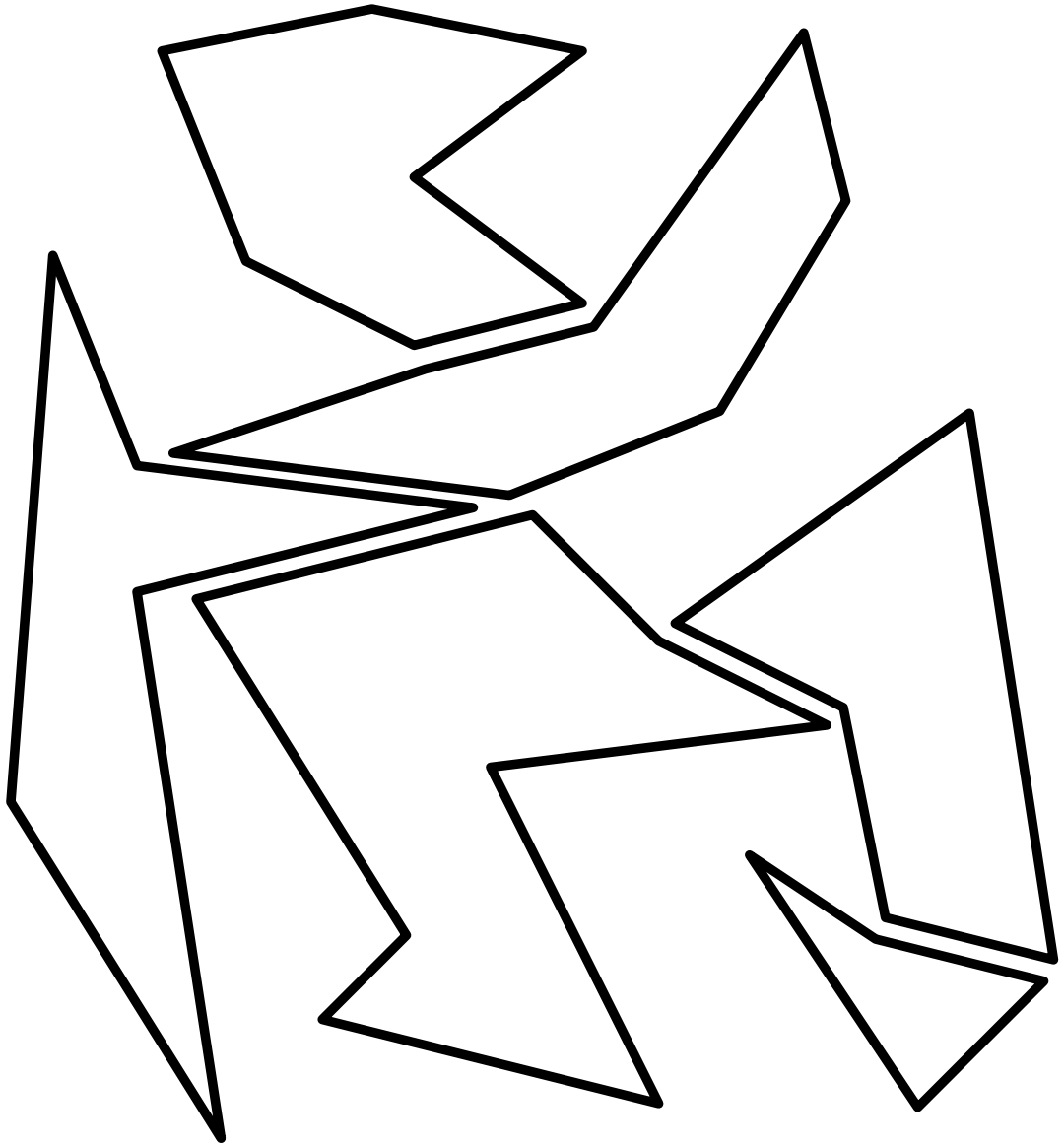


Sweep down, then up

From below:
Each polygon
separately!

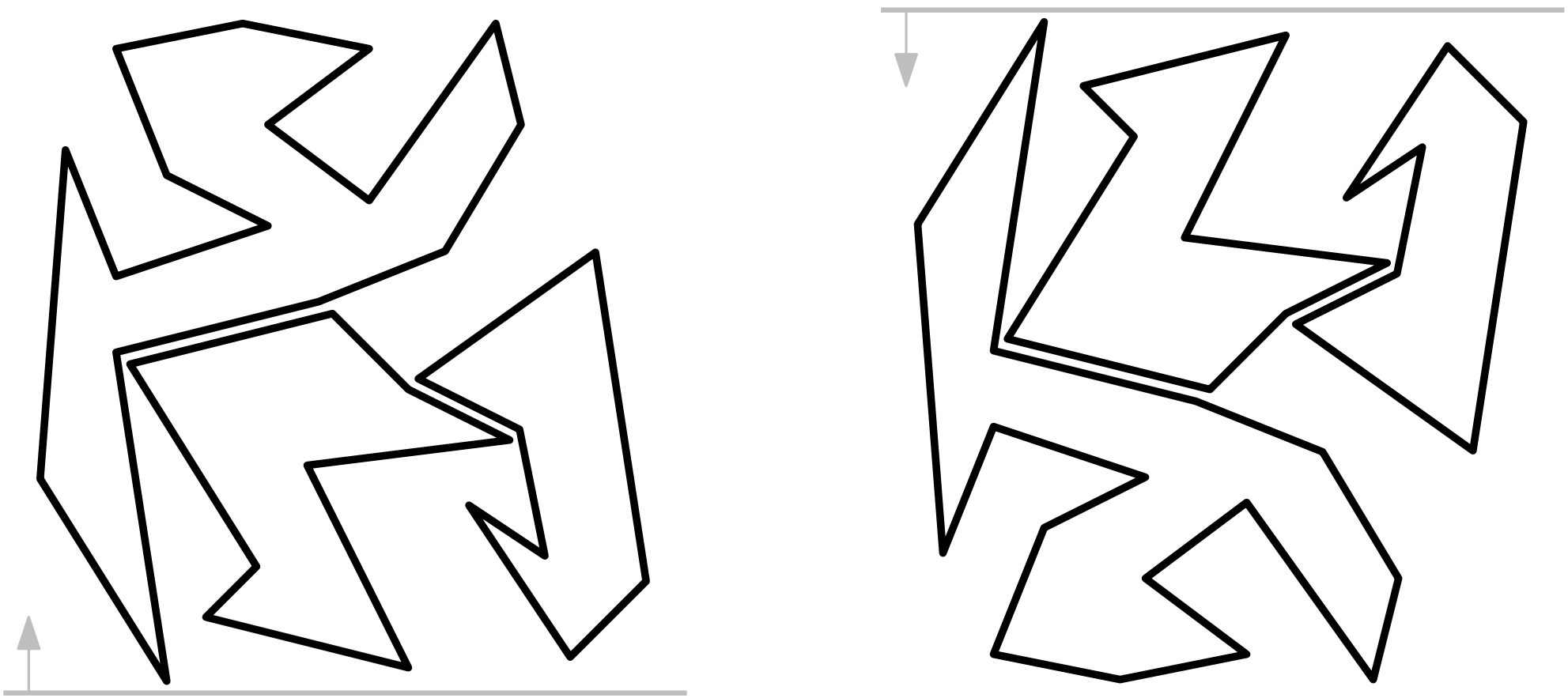


Sweep down, then up



Sweep down, then up

$$y \mapsto -y$$



Sweep down, then up

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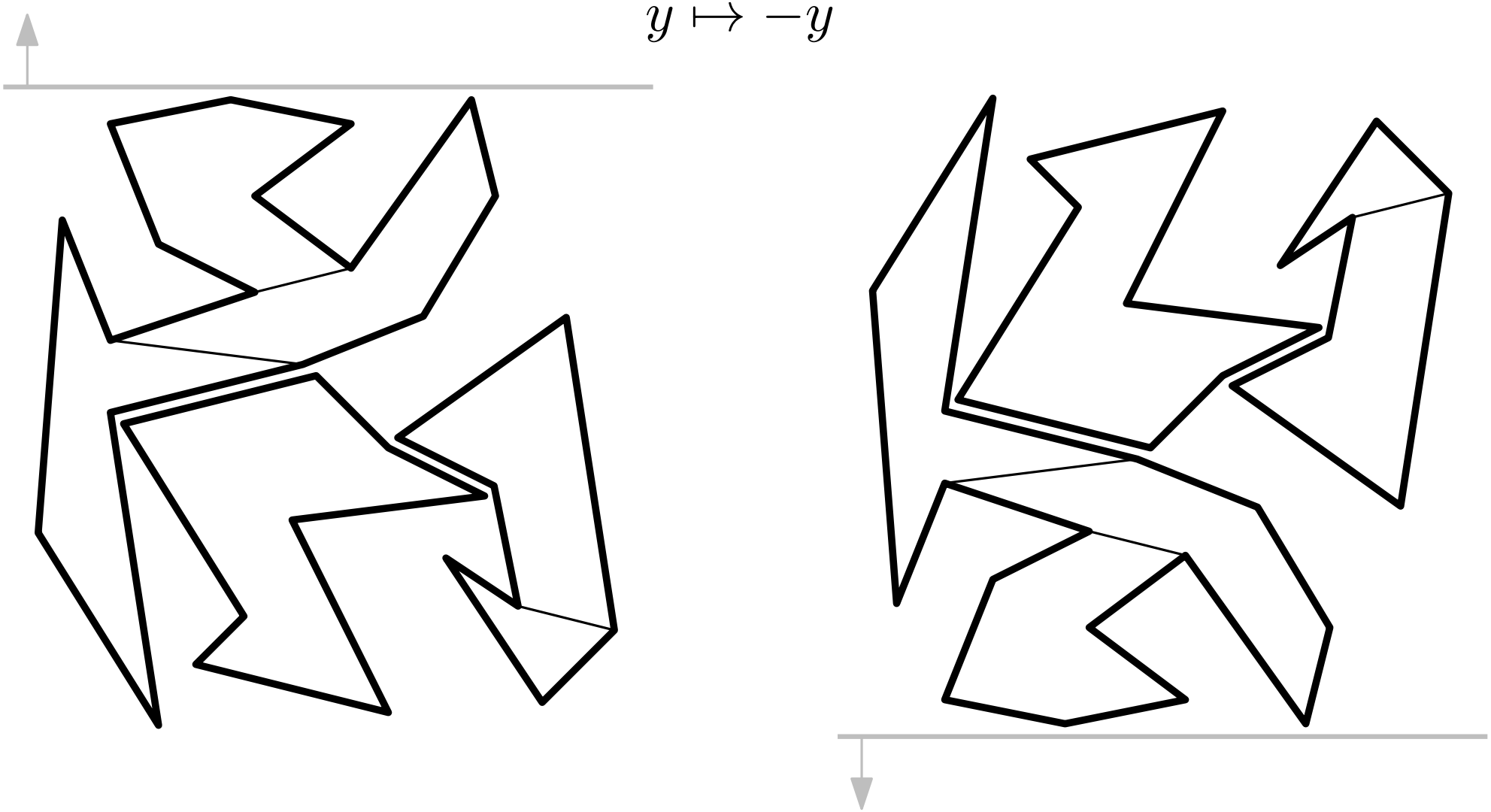
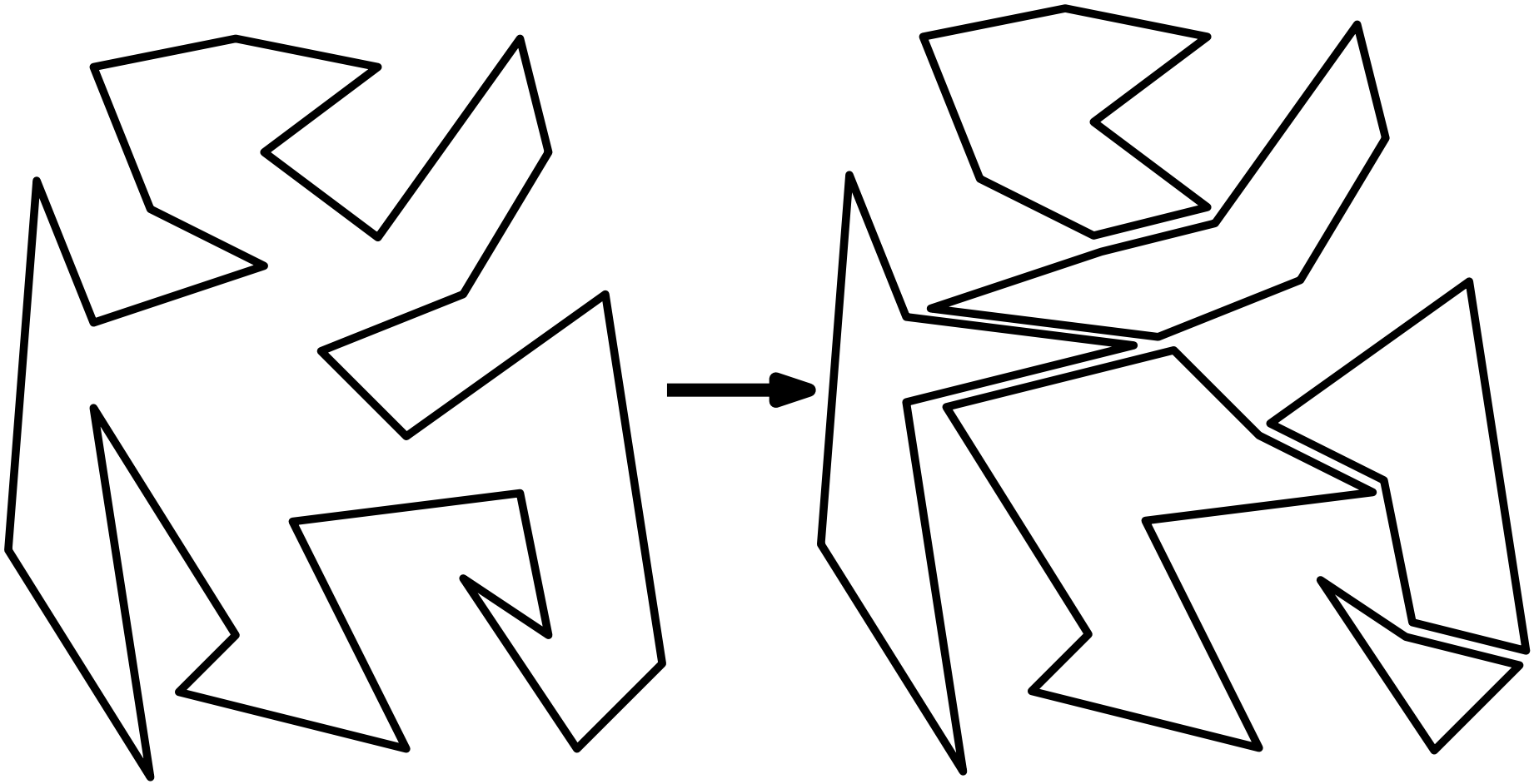


How many vertices?



n vertices

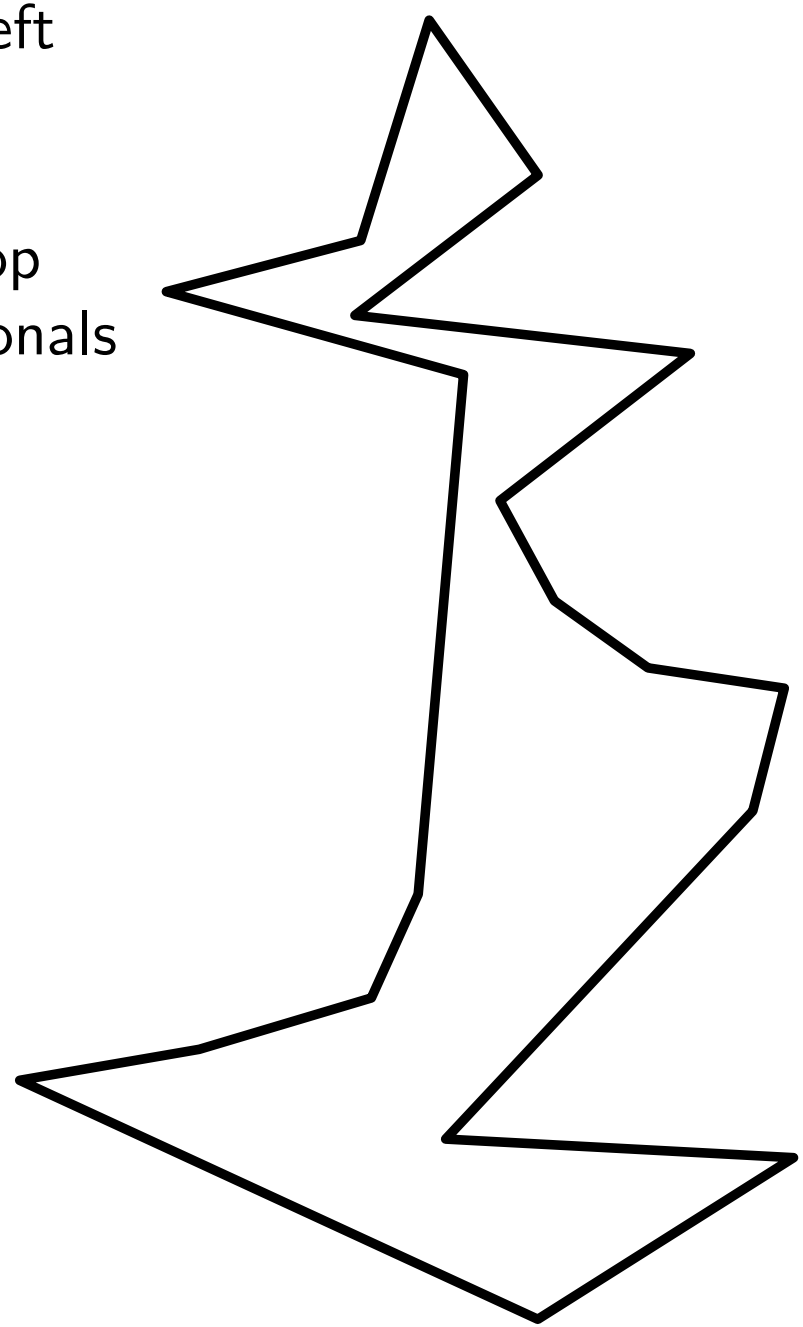
2 extra vertices per diagonal and
$\leq n-3$ diagonals $\Rightarrow$
$\leq n + 2(n-3) = 3n-6$ vertices

Efficiency

Sorting vertices by y-coordinates: $O(n \log n)$.
Handling n events: $O(n \log n)$ time.
Doing it again from below (at most $3n - 6$ vertices now).
In total $O(n \log n)$.

Triangulating a y-monotone polygon

1: Merge vertices of left and right chain to get sorted order
2: Traverse vertices top down and create diagonals to all possible vertices above



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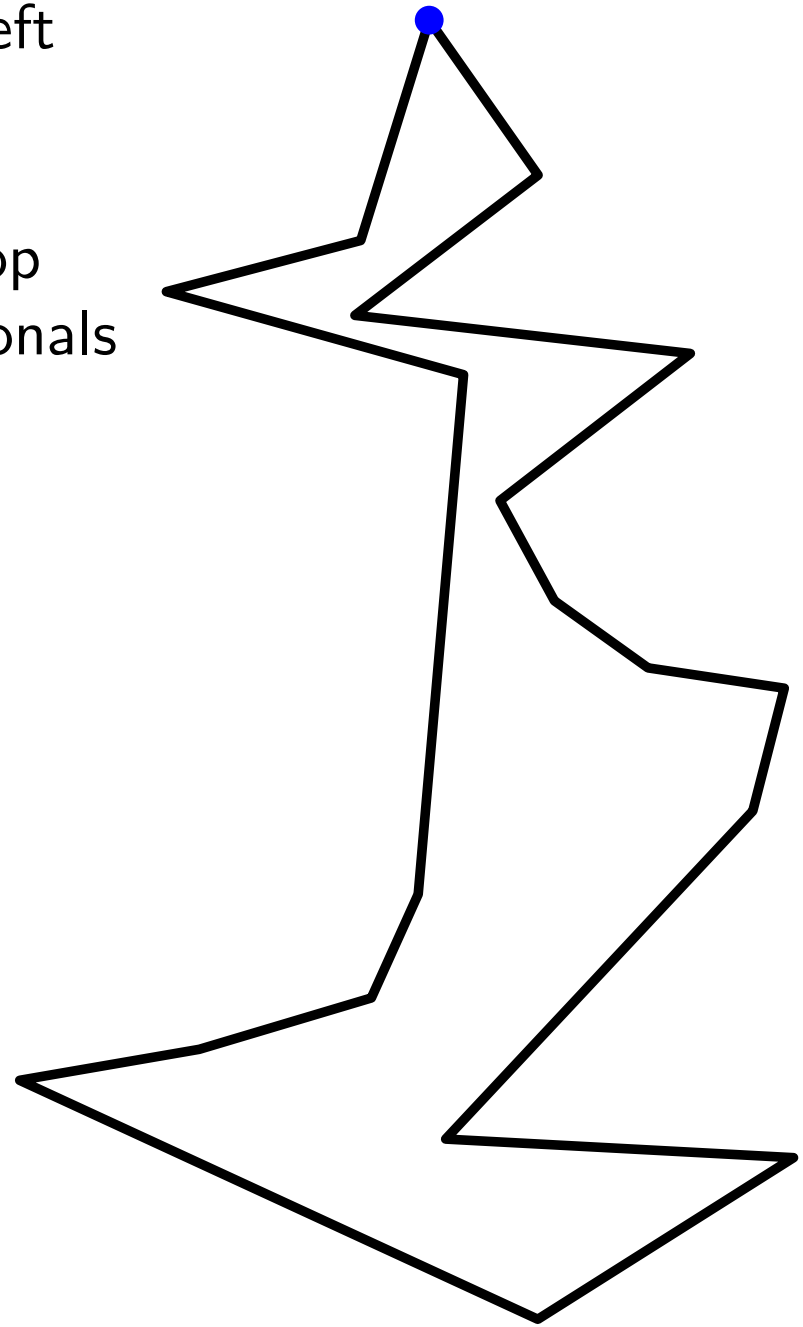

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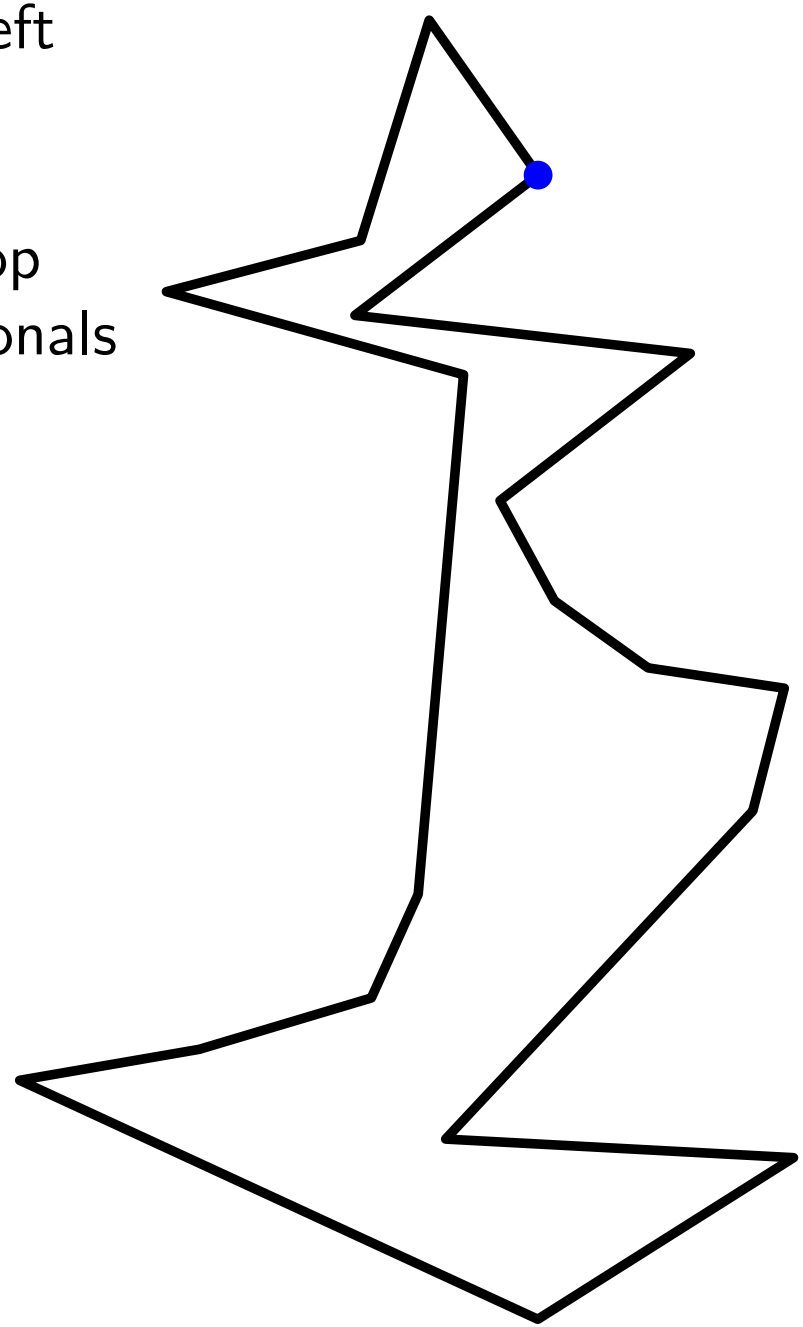

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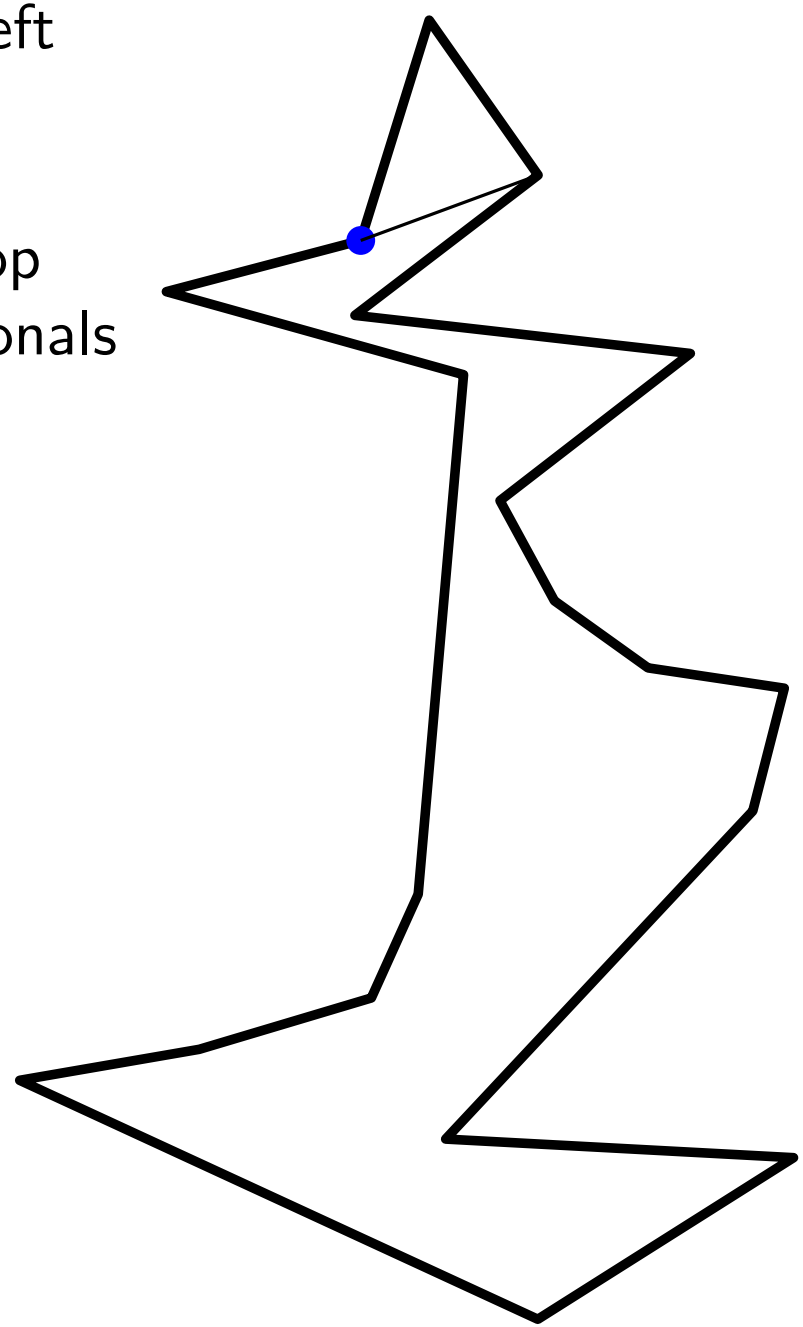

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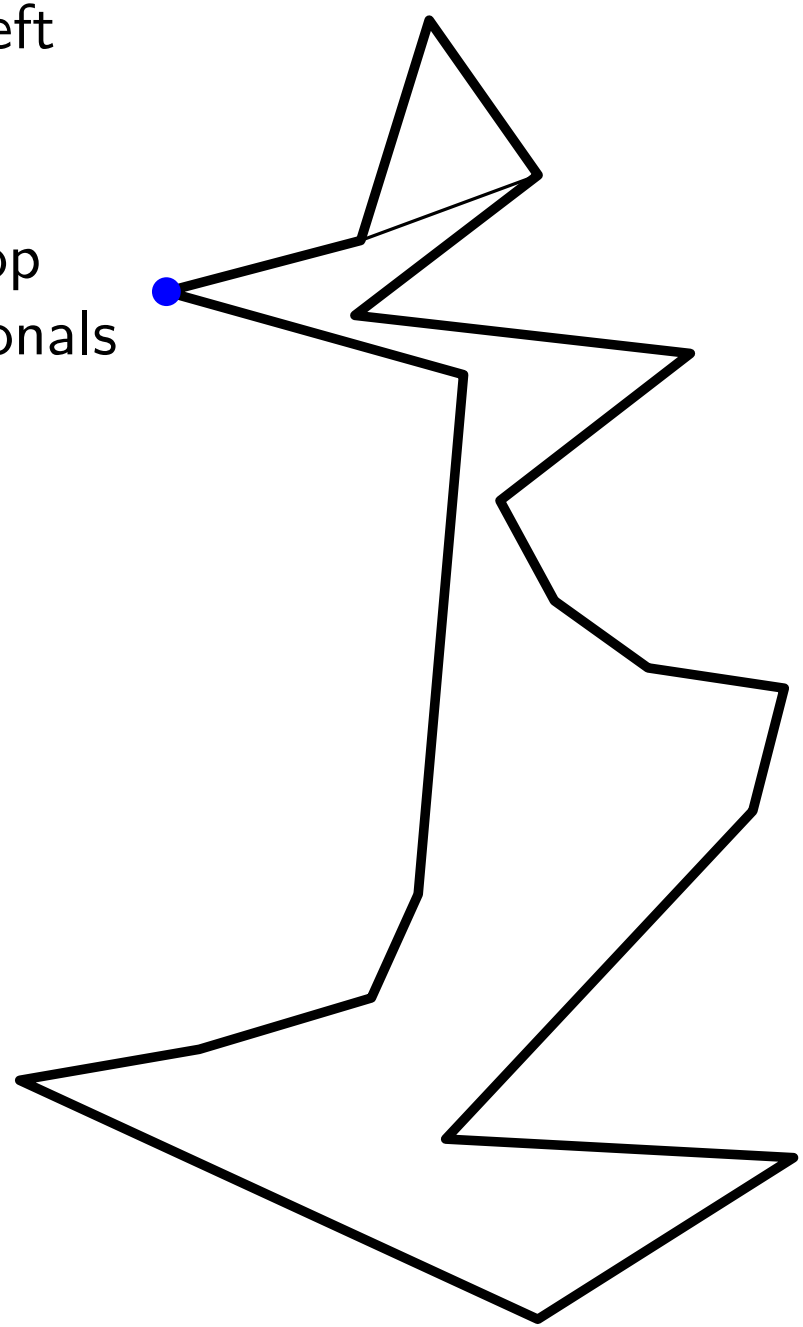

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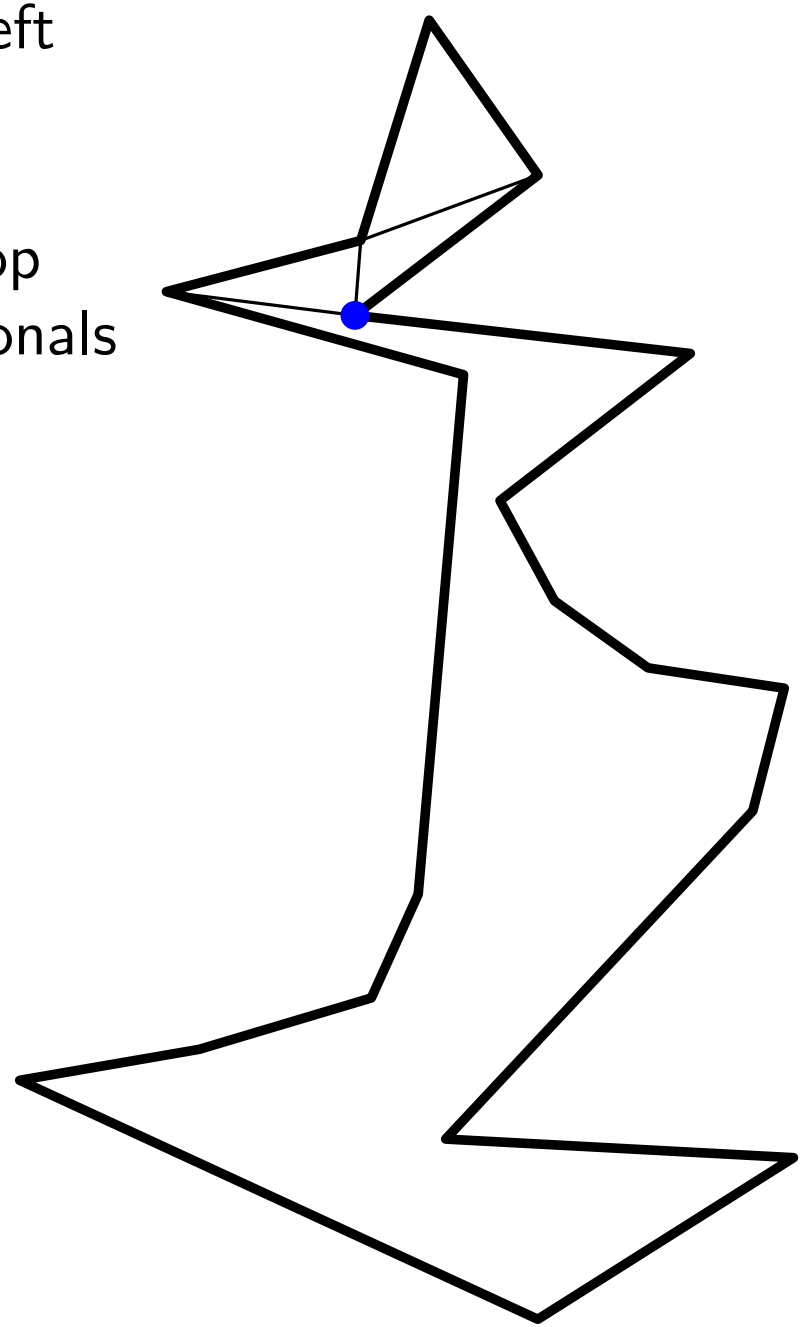

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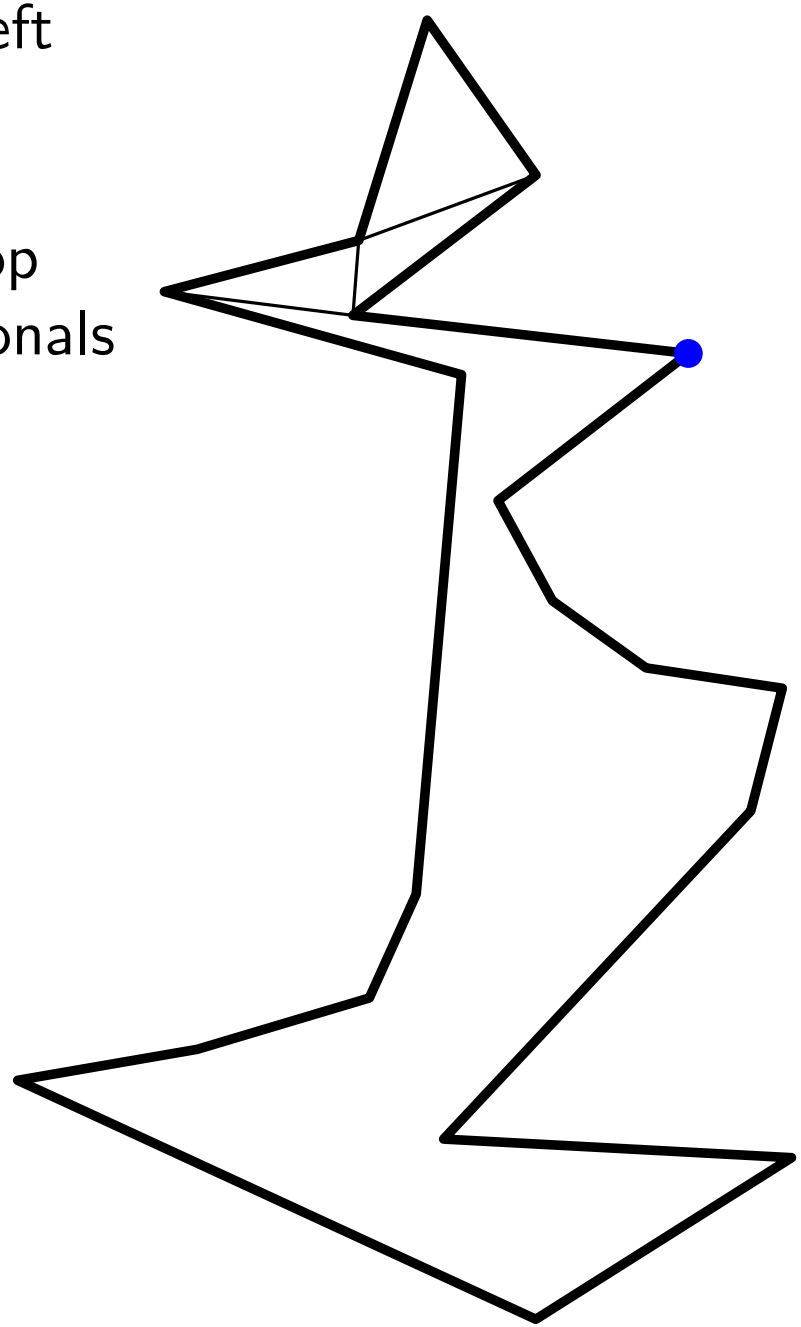

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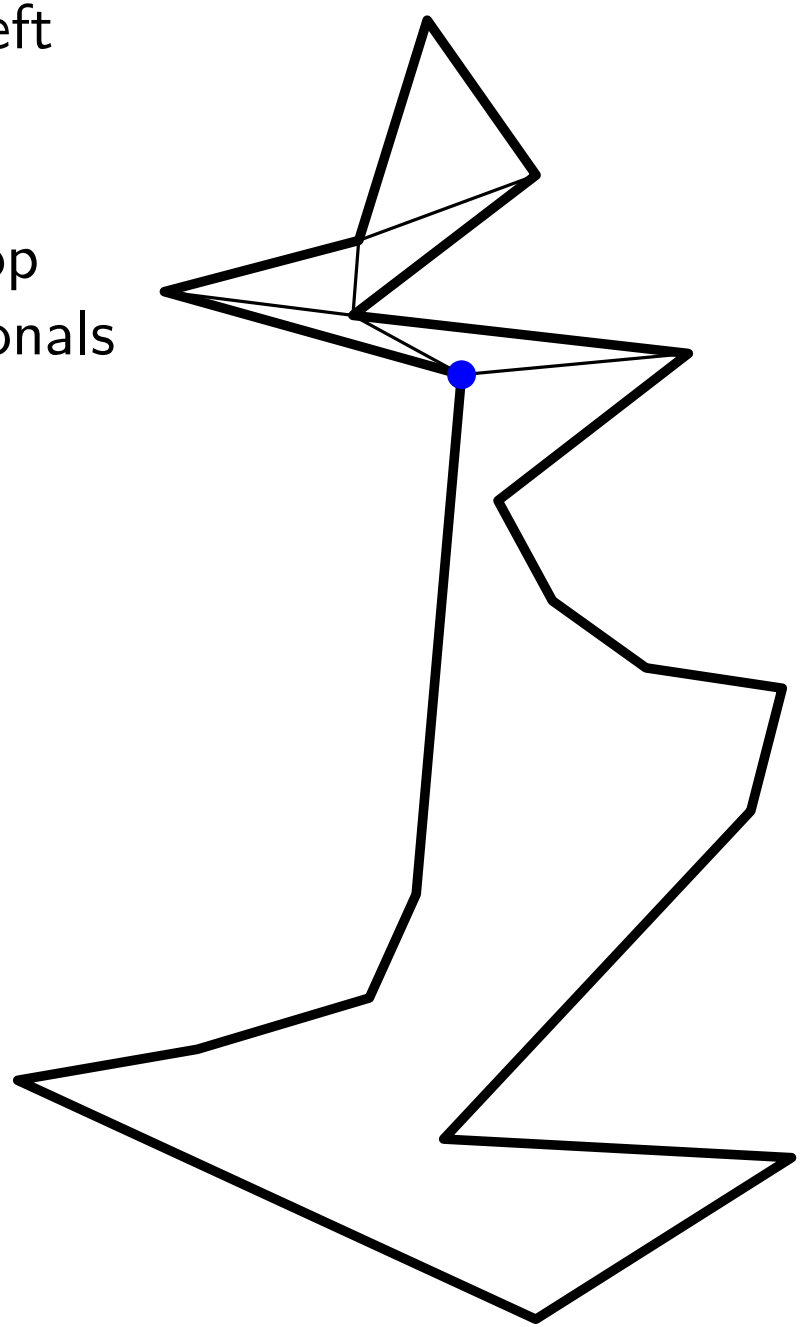

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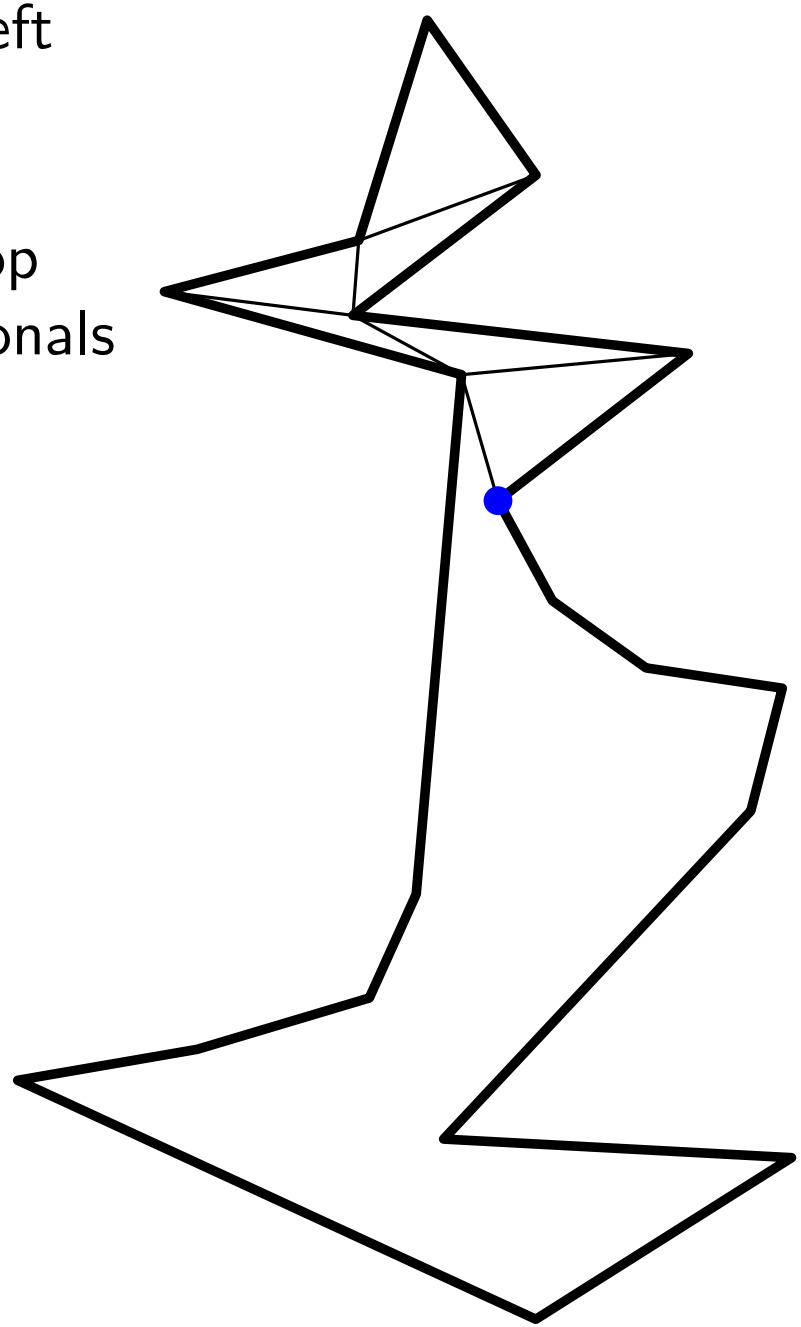

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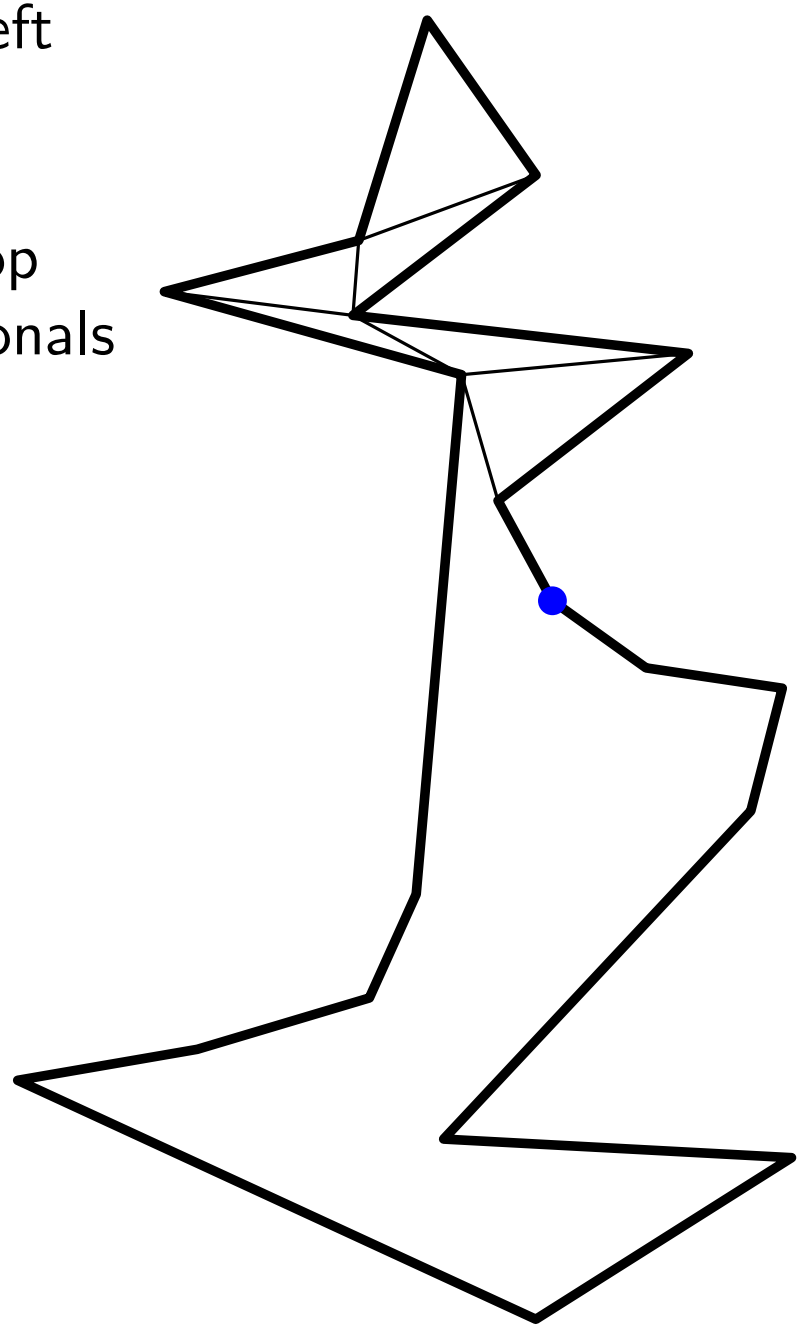

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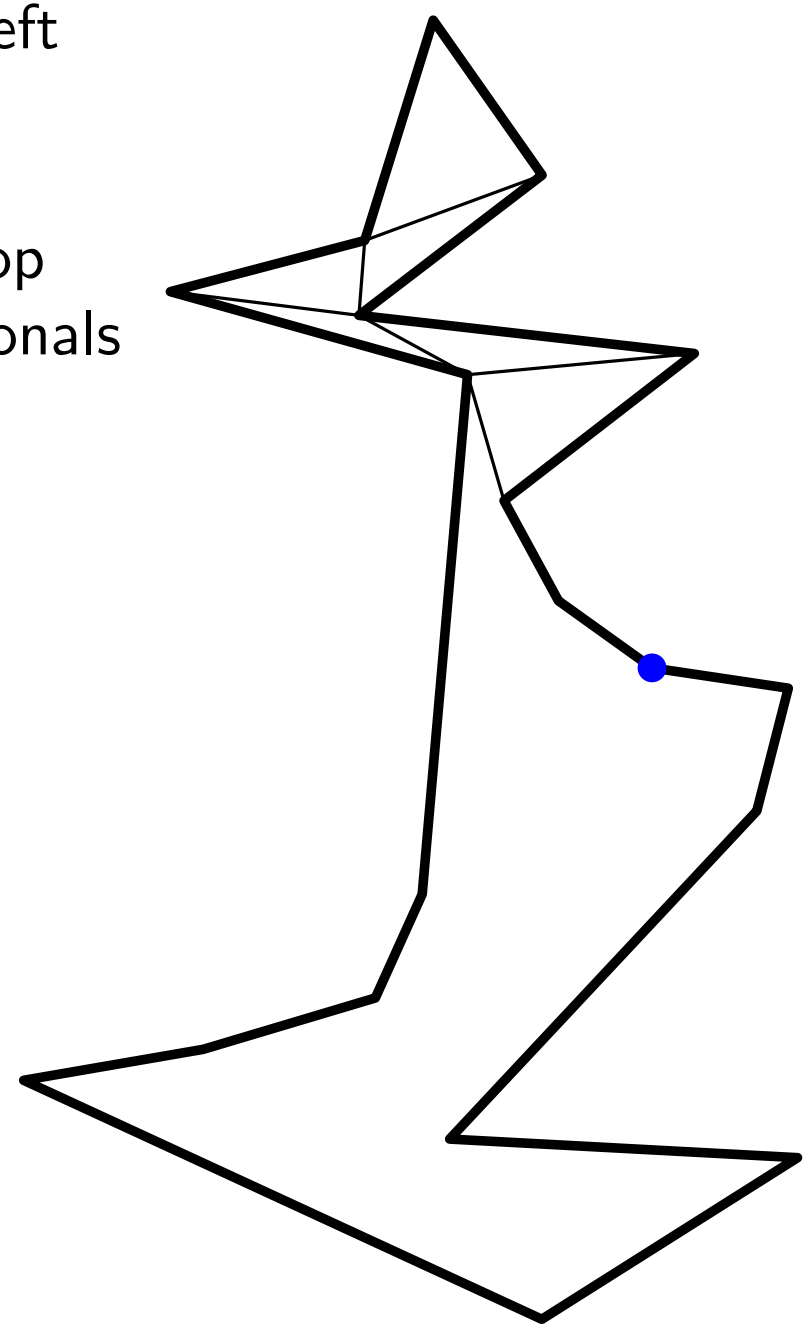

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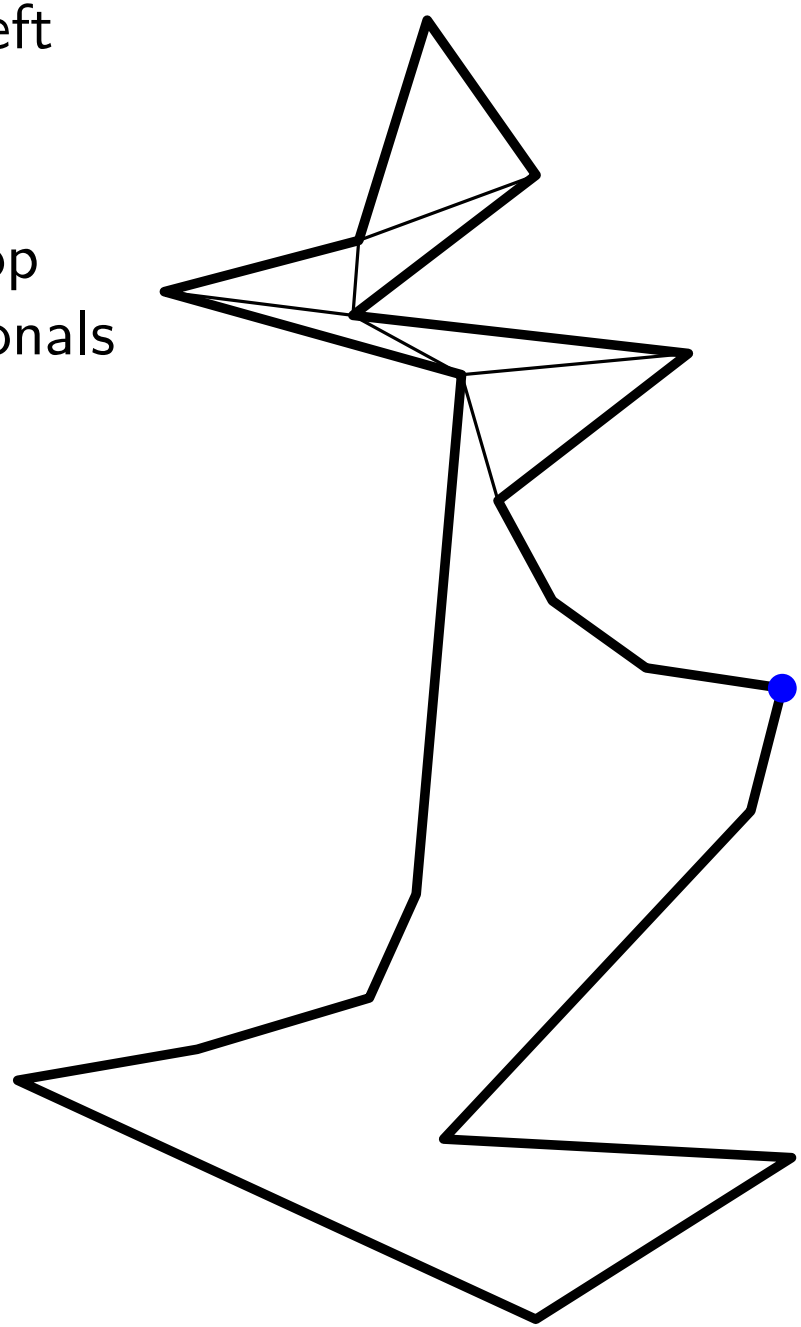

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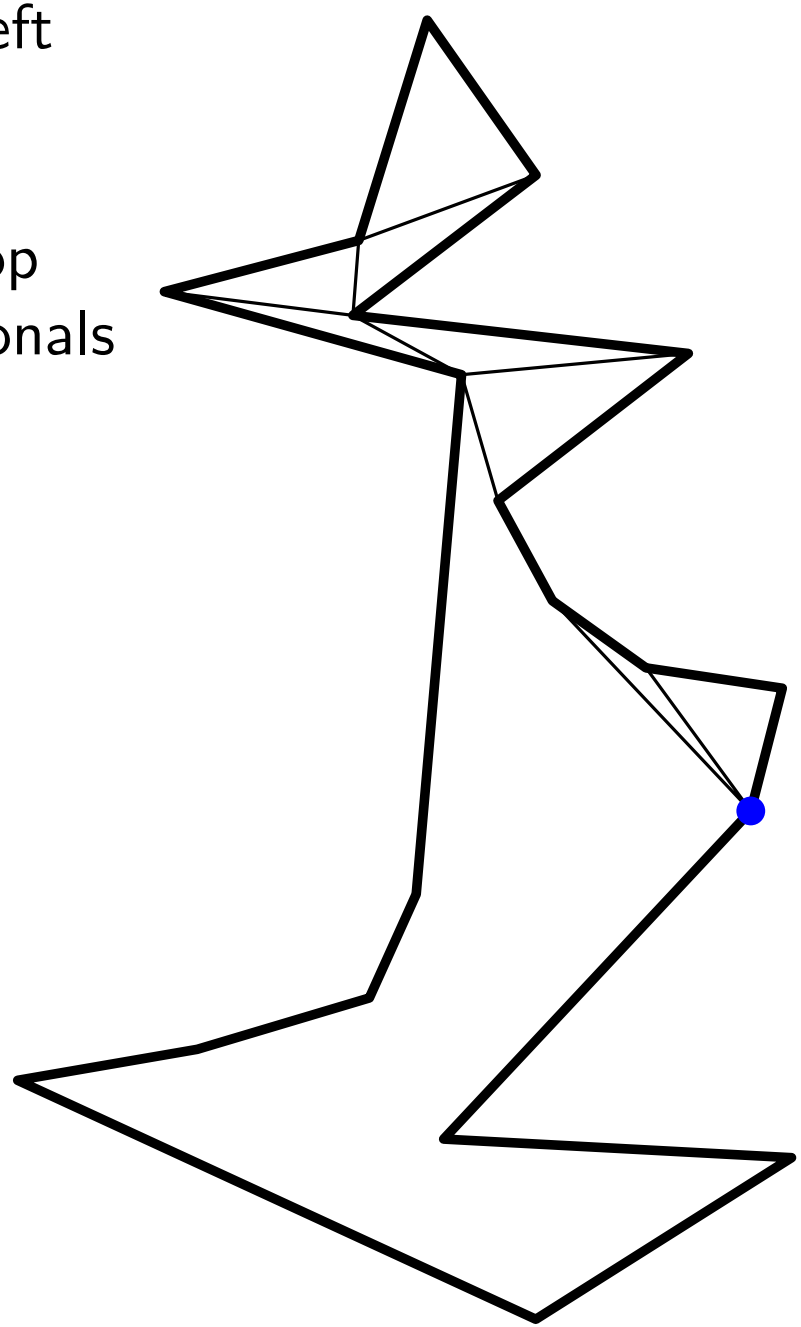

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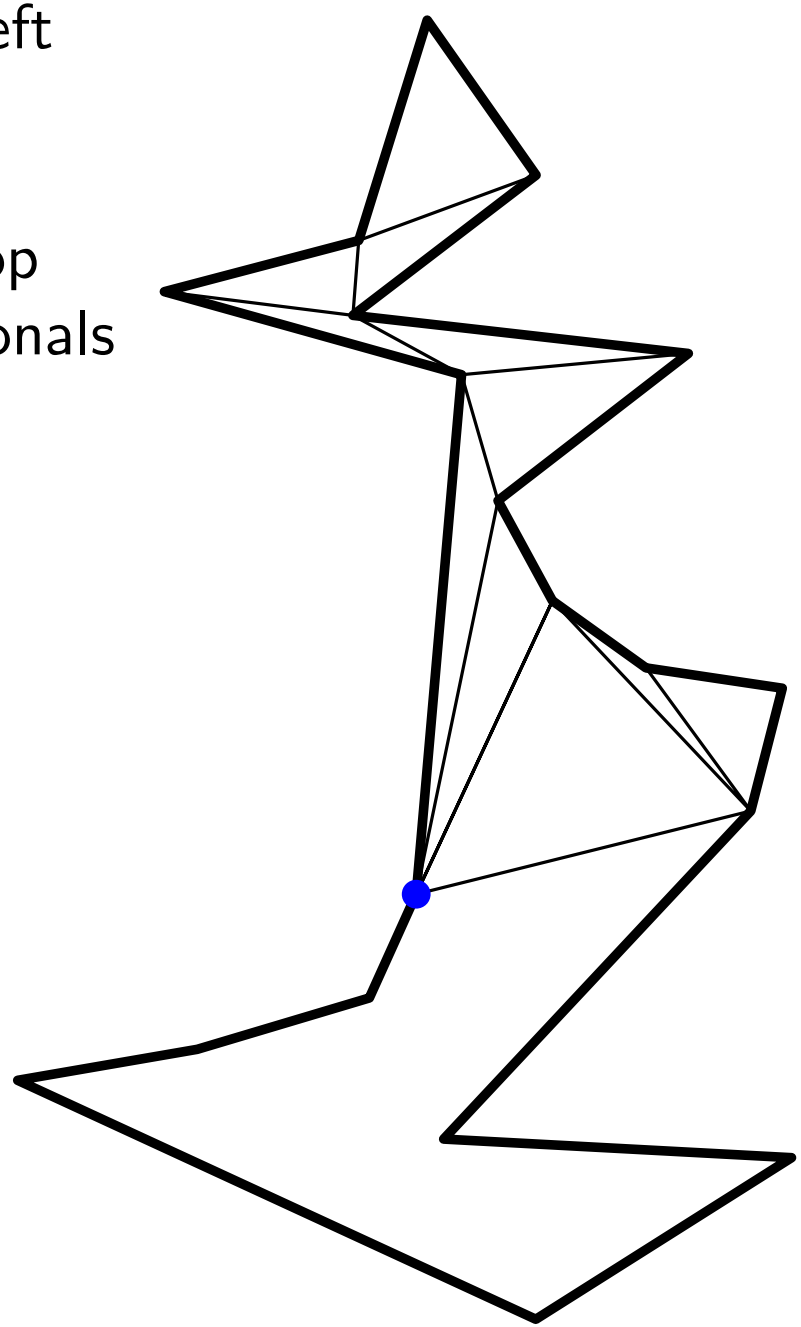

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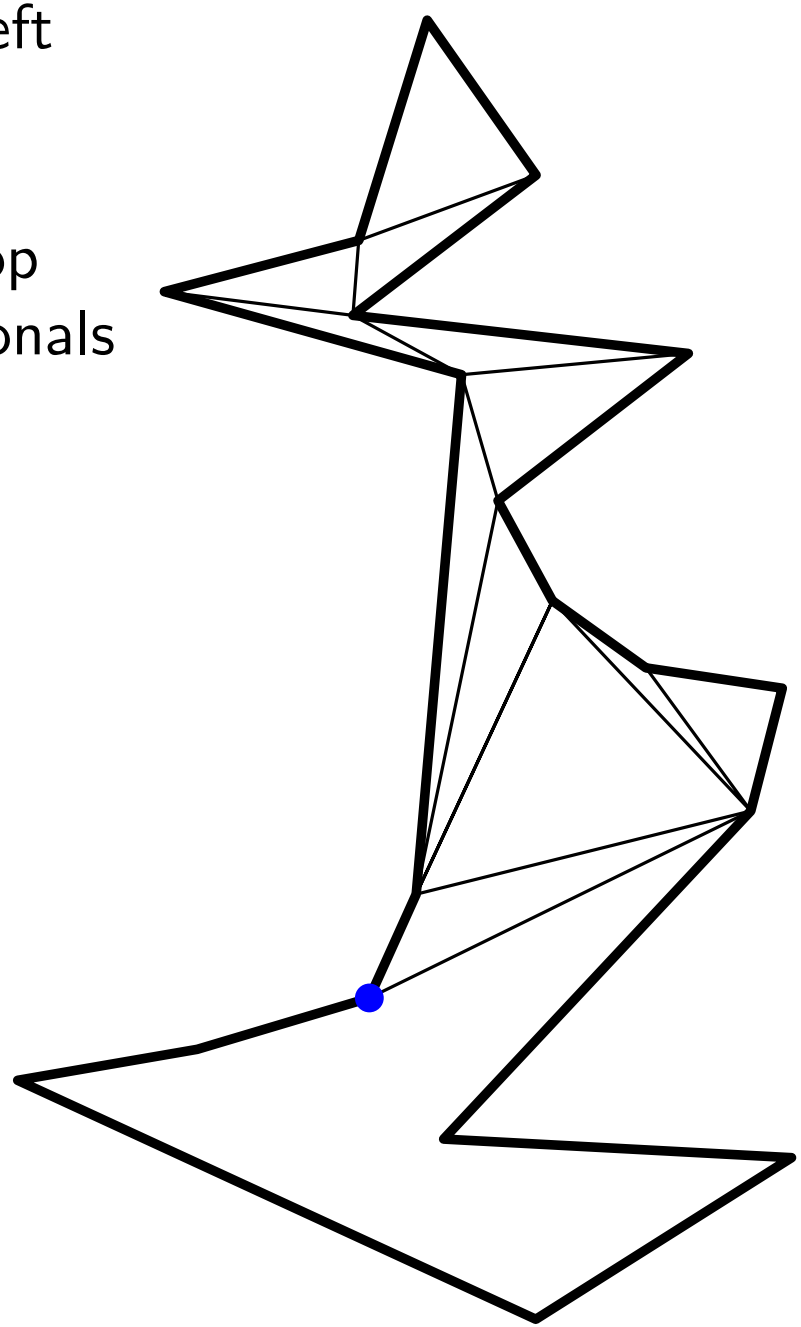

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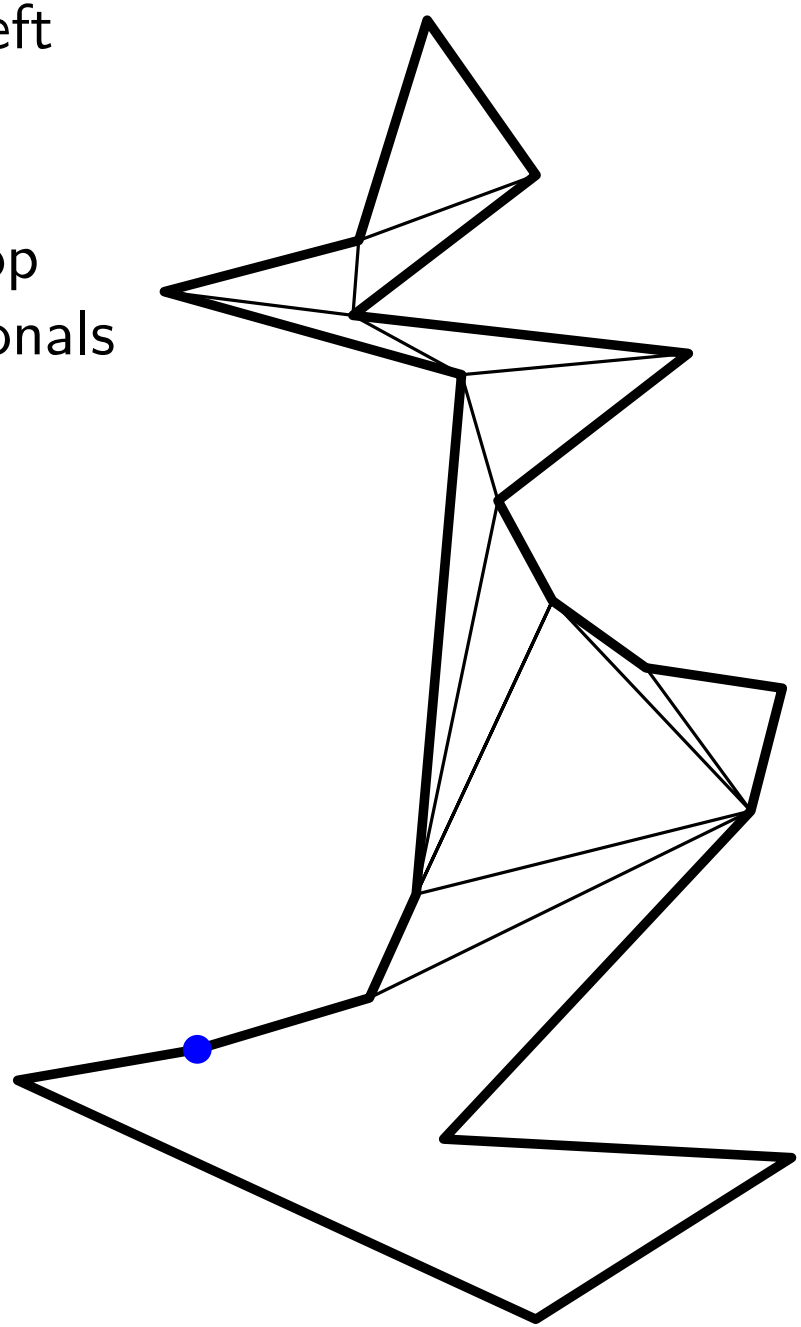

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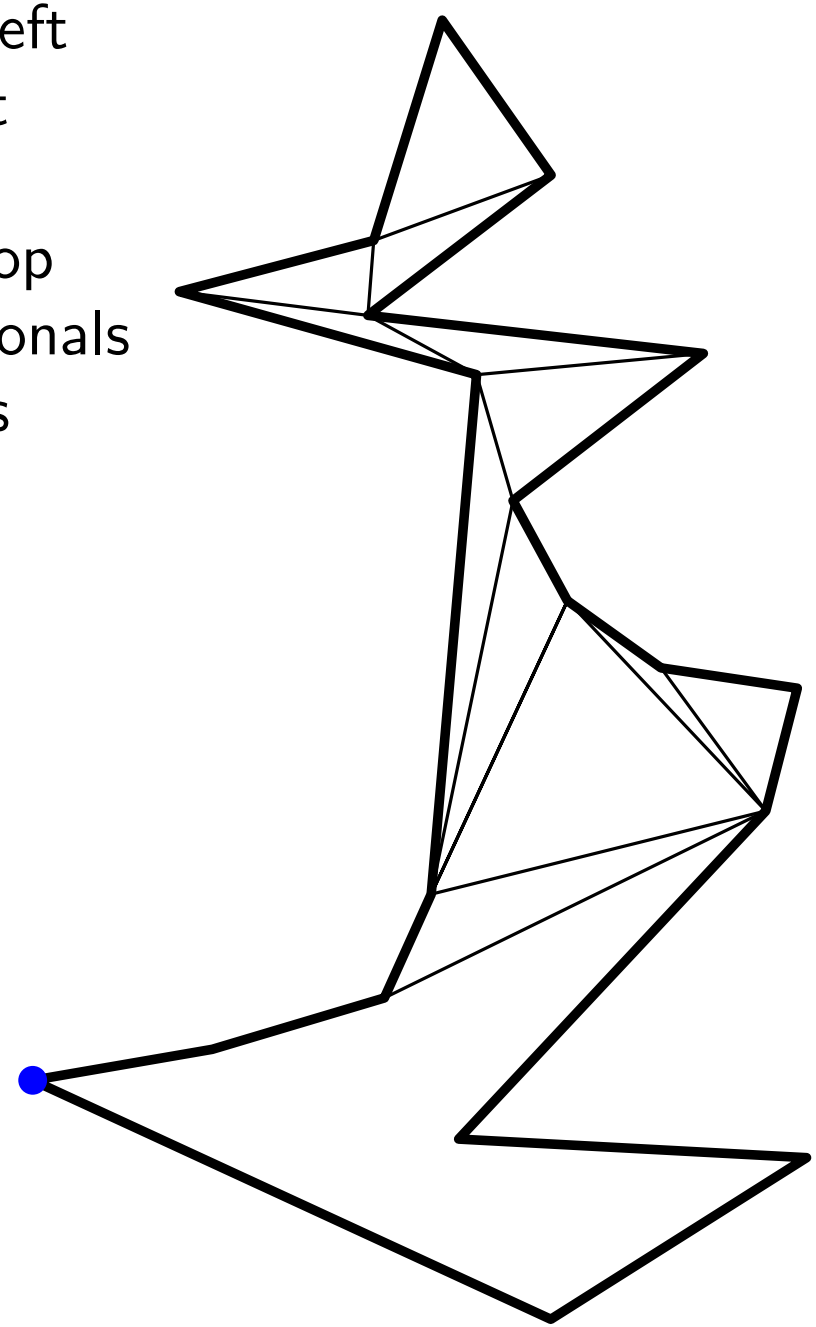

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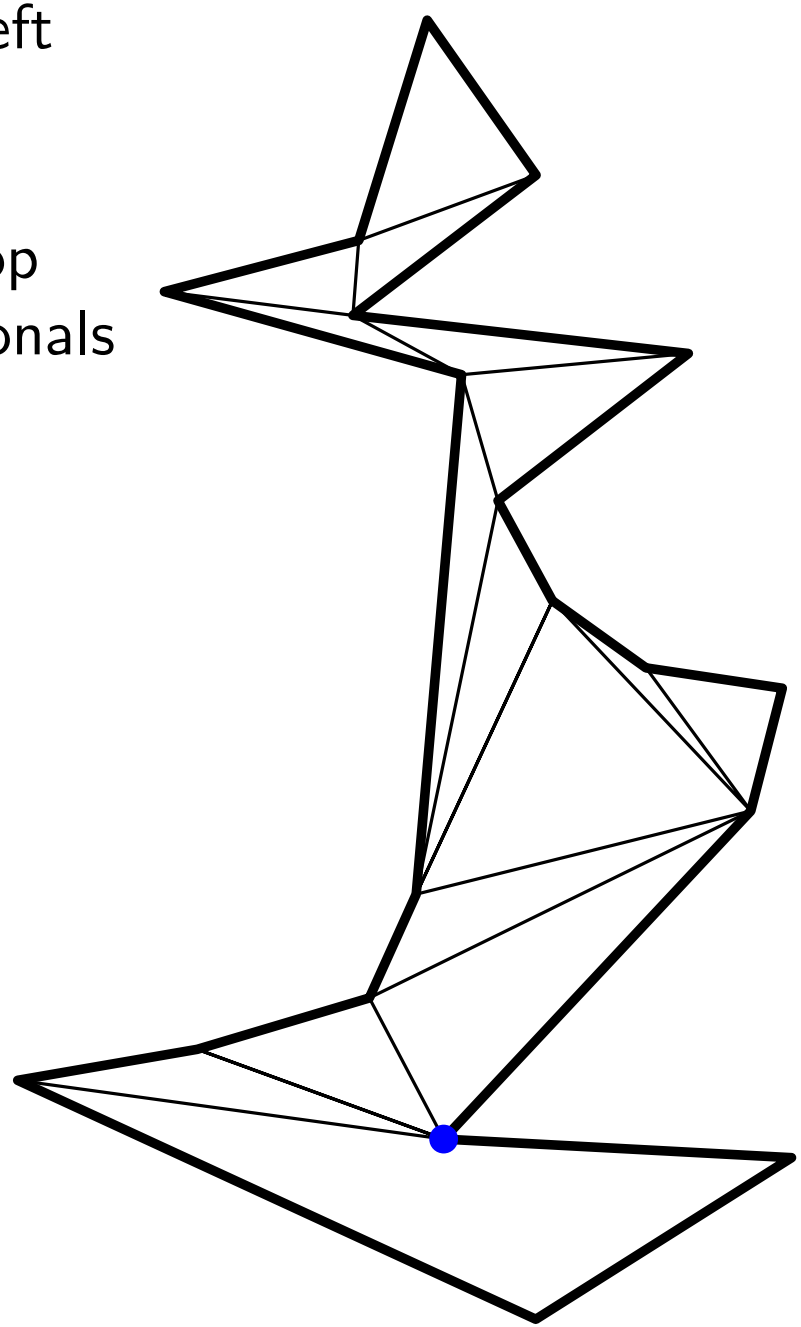

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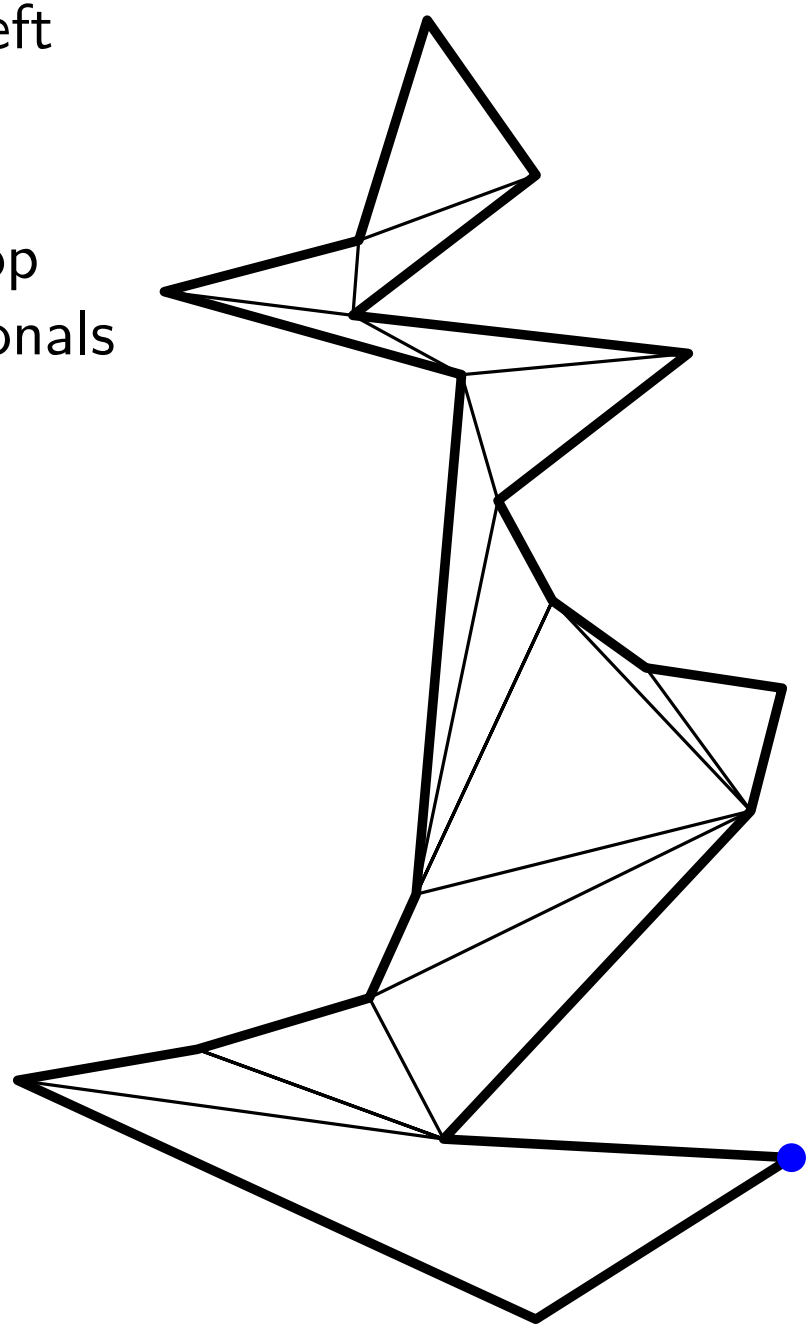

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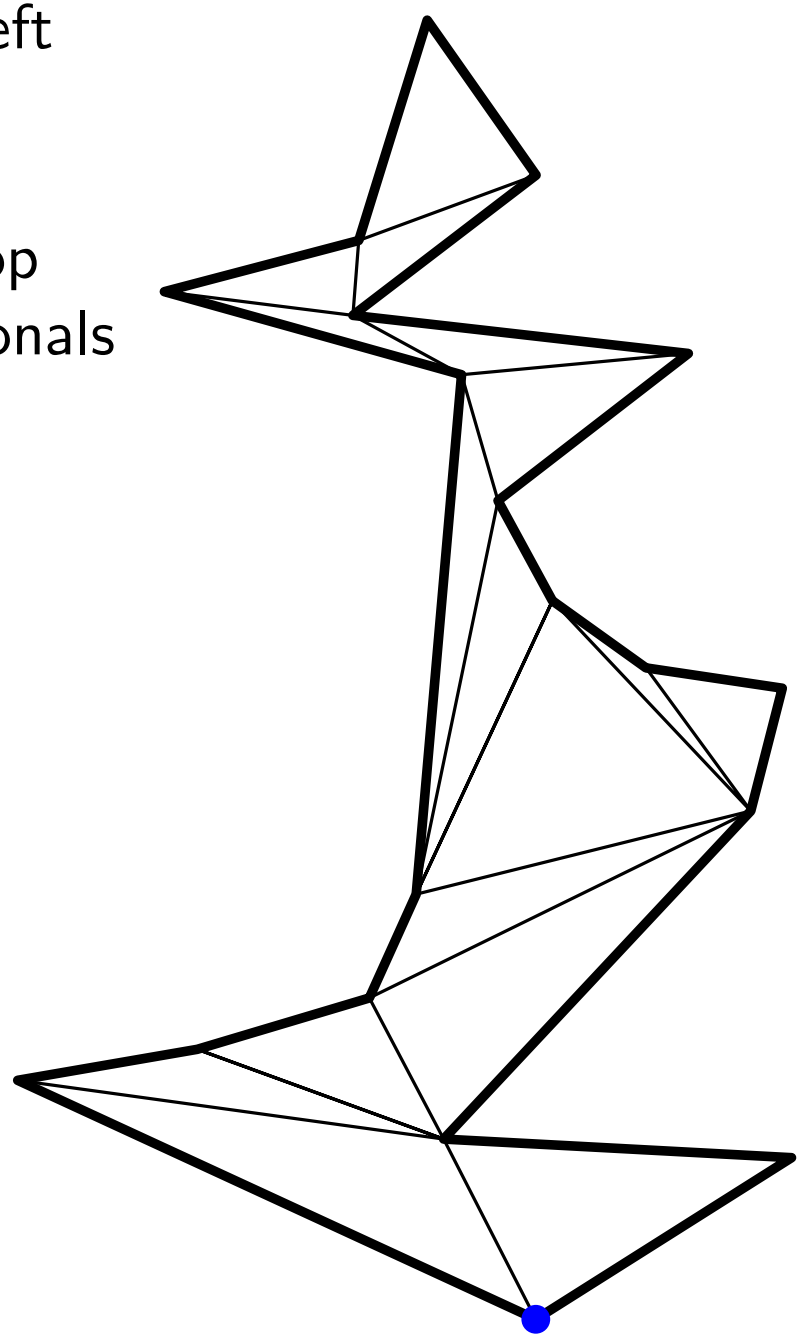
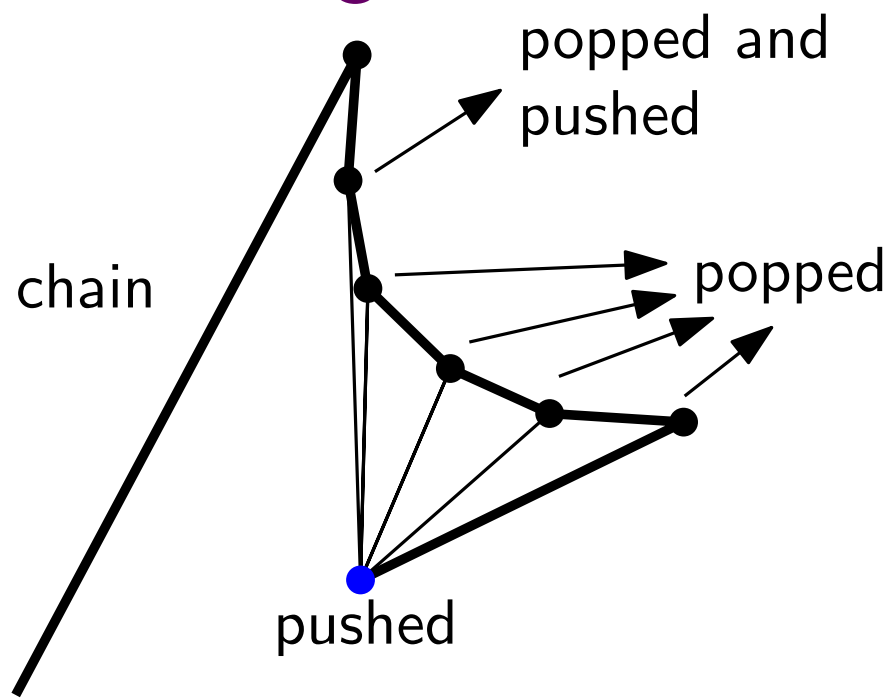
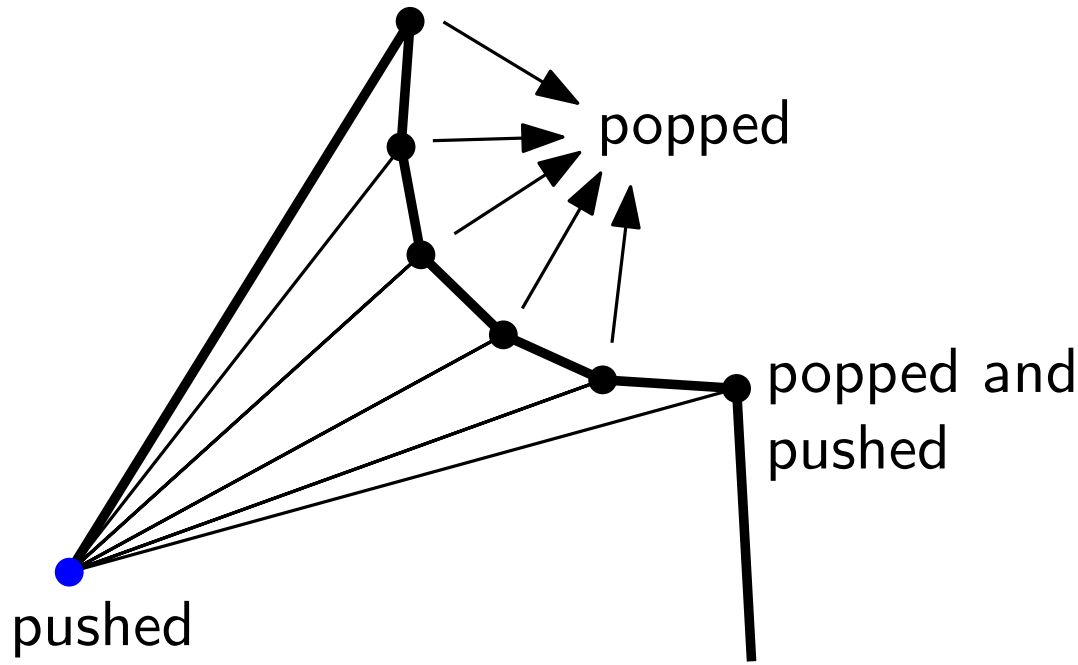


Using a stack to make diagonals

Case 1: New vertex on the same chain (here right).



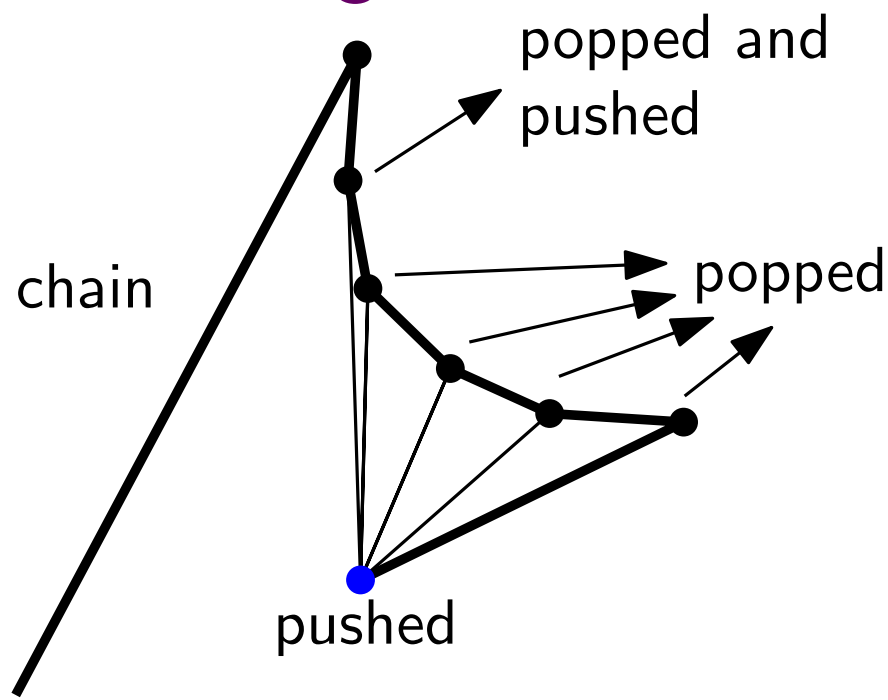
Case 2: New vertex on the other chain.



Using a stack to make diagonals

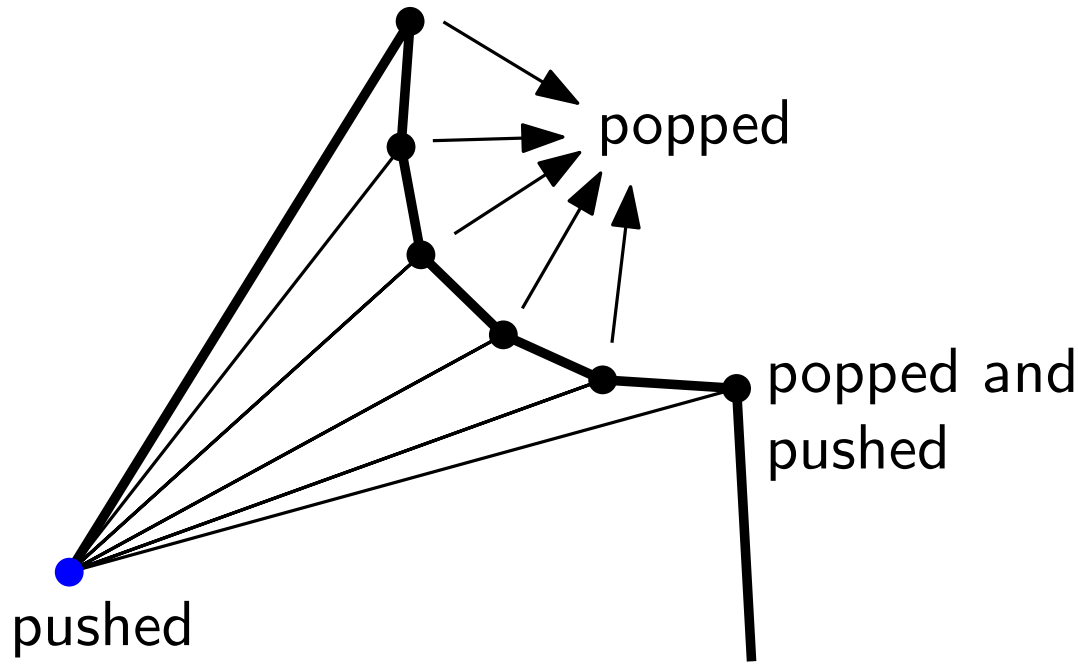
Case 1: New vertex on the same chain (here right).

edge on the left chain

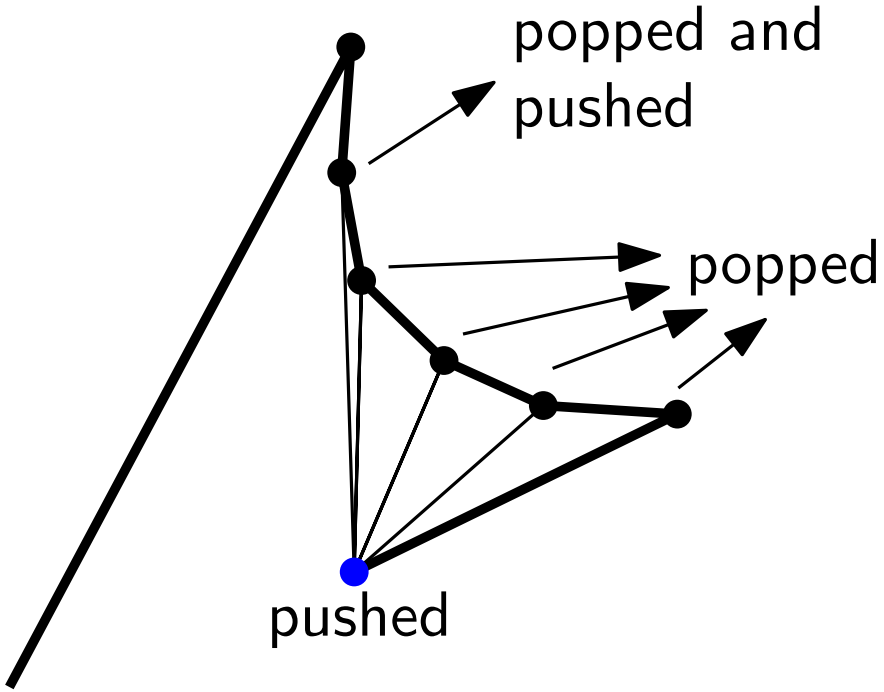


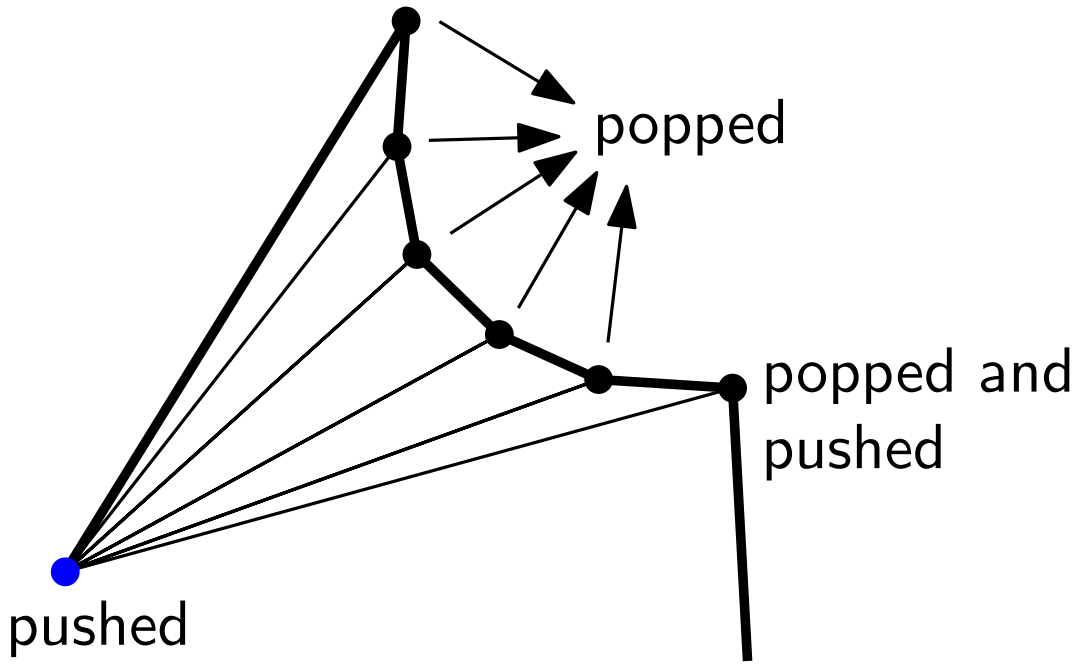
Case 2: New vertex on the other chain.

Why are all diagonals OK?



Time complexity



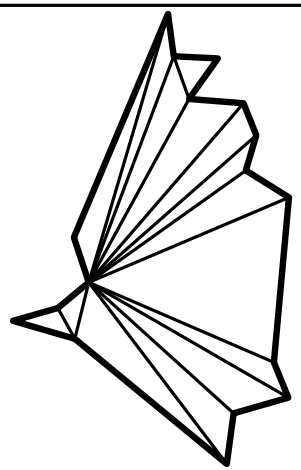


Note: In either case, we push two vertices $\Rightarrow \leq 2n$ pushes, pops and diagonal checks.

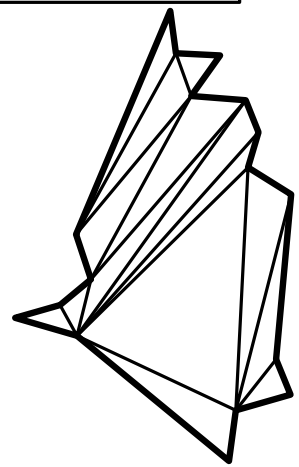
Time complexity: $O(n)$ for merging and traversing.
n: number of vertices of this y-monotone polygon

How does the triangulation look?

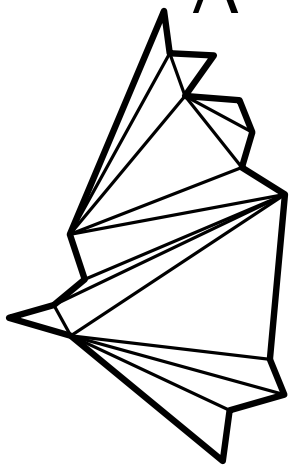
socrative.com → Student login,
Room name: ABRAHAMSEN3464



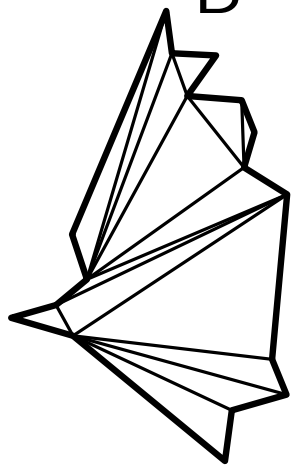
A



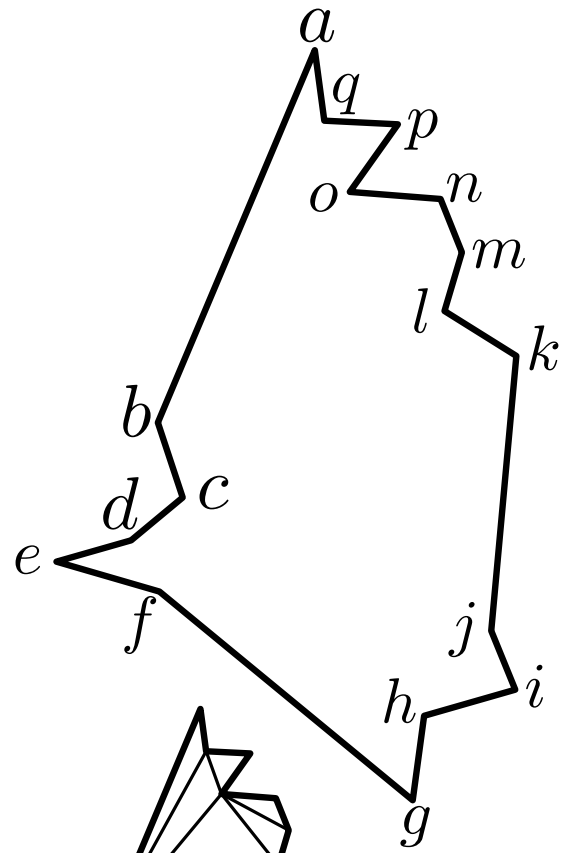
B

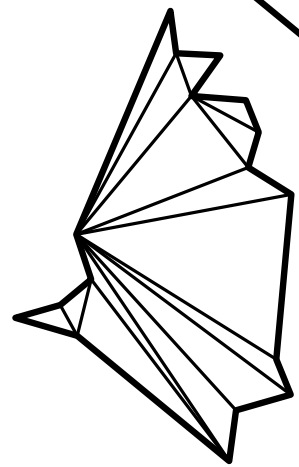


C



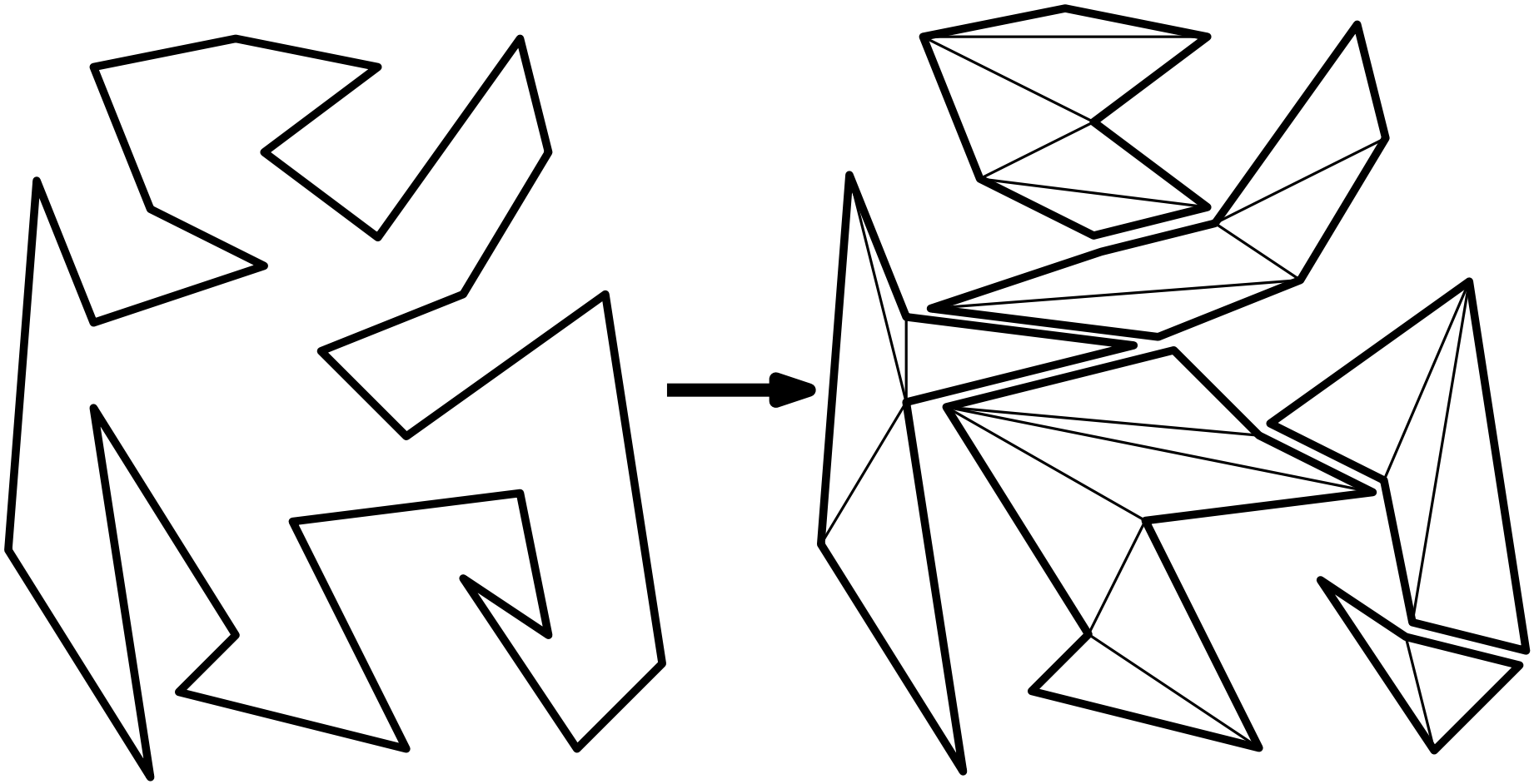
D





E

Total time complexity



$\leq 3n - 6$ vertices $\Rightarrow O(n)$ time to triangulate monotone polygons.
In total $O(n \log n)$ time
(monotone partition dominates).

History of Triangulation Algorithms

1979: Garey, Johnson, Preparata, Tarjan. $O(n \log n)$ sweep-line algorithm, similar to this.
1982: Chazelle. $O(n \log n)$ divide-and-conquer algorithm.
1986: Tarjan and Van Wyk. $O(n \log \log n)$ algorithm.
1988: Clarkson, Tarjan, and Van Wyk. Randomized $O(n \log^* n)$ algorithm. Two other algorithms with same running time around the same time.
1990: Chazelle. Optimal $O(n)$ algorithm.

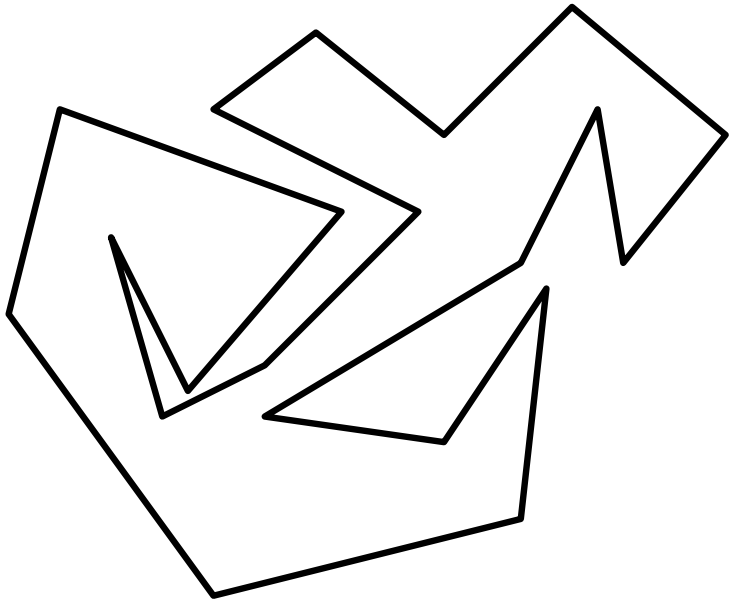
$\log^* n$: number of times to apply $\log$ before we get to ≤ 1.

$$\log^* n = \begin{cases} 0, & \text{if } n \leq 1, \\ 1 + \log^*(\log n), & \text{if } n > 1. \end{cases}$$

n	$\log^* n$
$(-\infty, 1]$	0
$(1, 2]$	1
$(2, 4]$	2
$(4, 16]$	3
$(16, 65536]$	4
$(65536, 2^{65536}]$	5

Use of Triangulations

Visibility problems.



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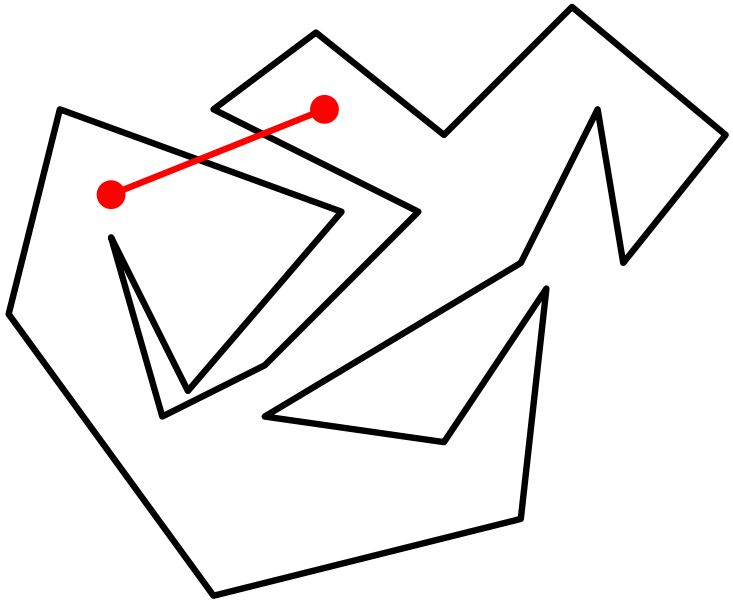

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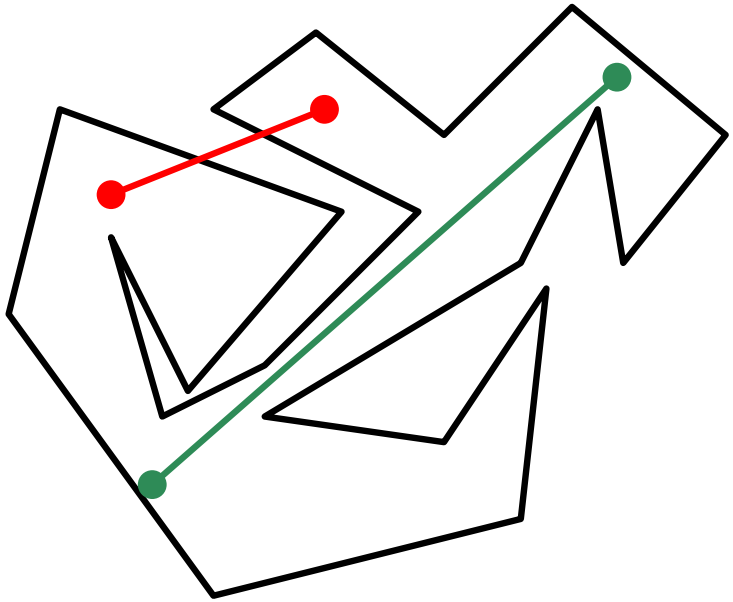
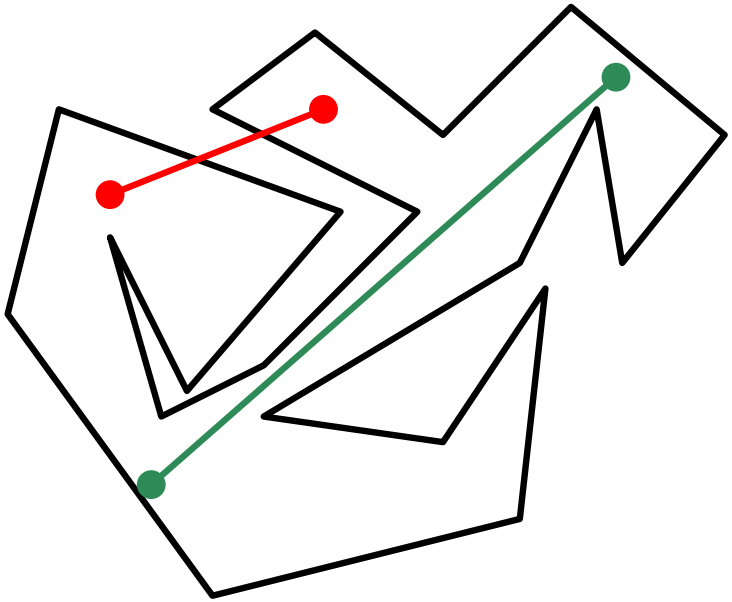
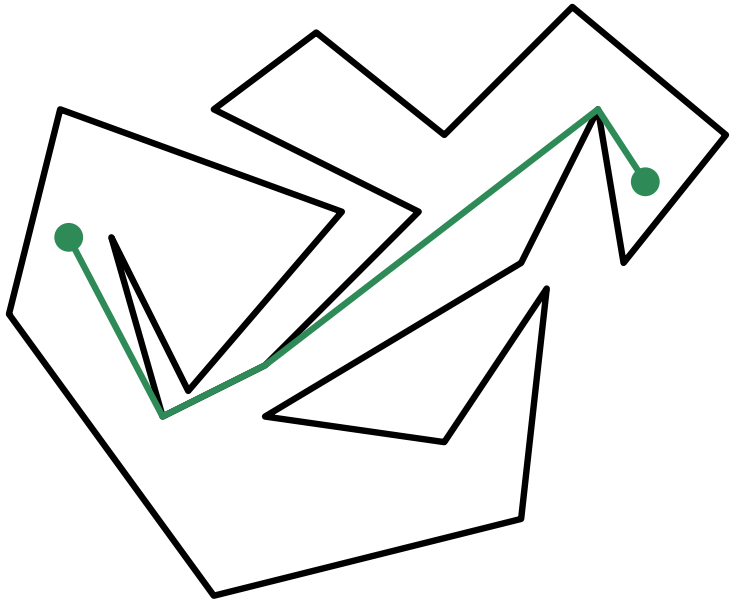


Use of Triangulations

Visibility problems.

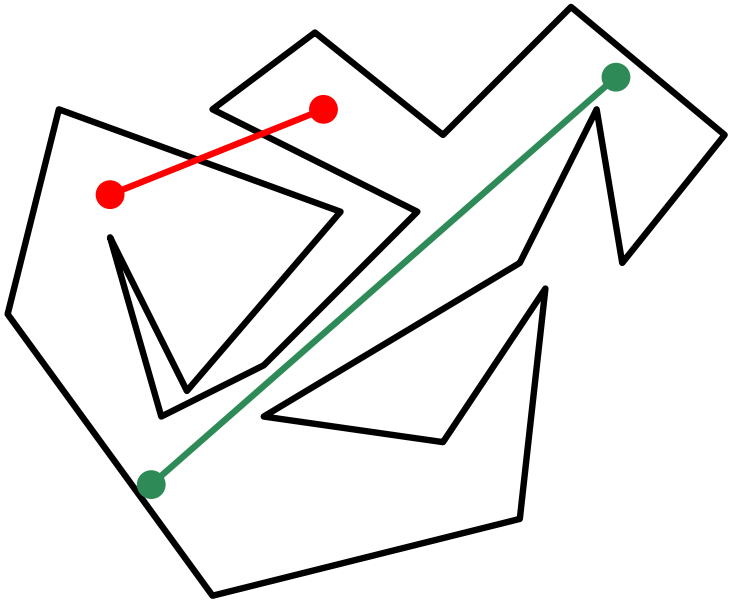


Shortest paths.



Use of Triangulations

Visibility problems.



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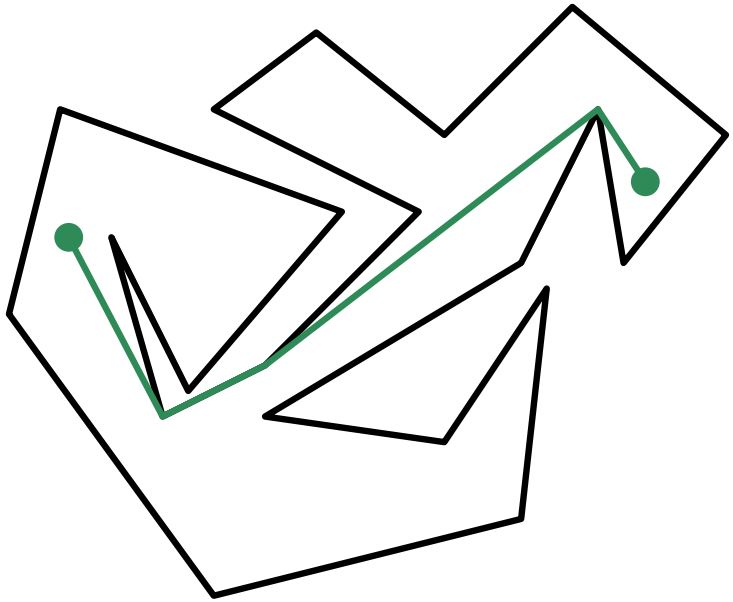

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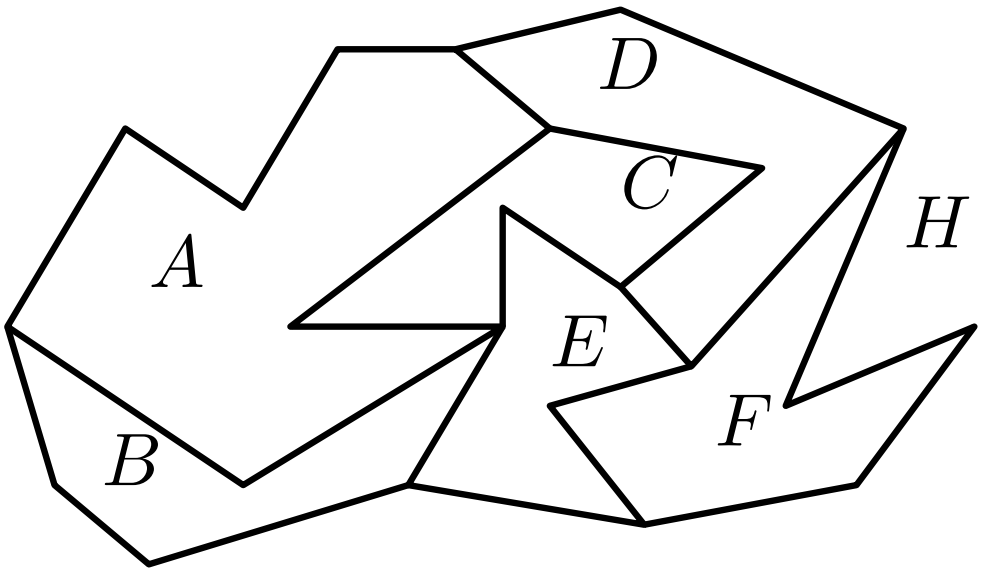

Use of Triangulations

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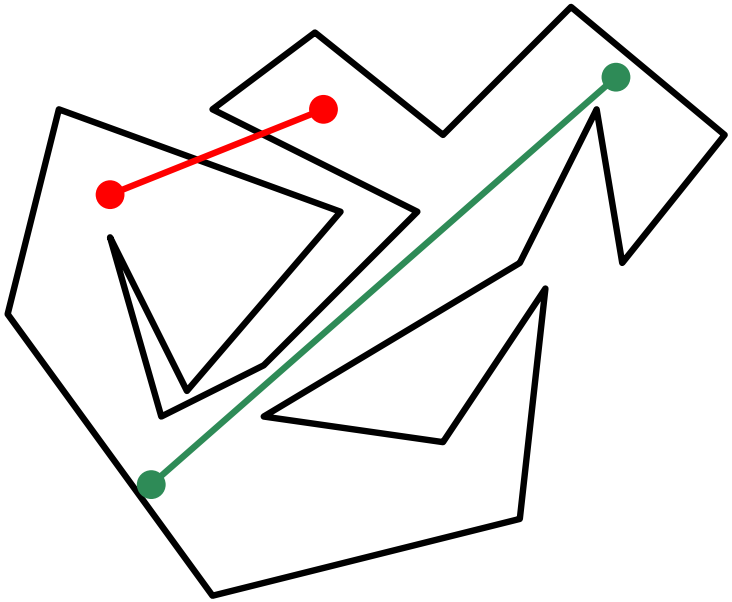

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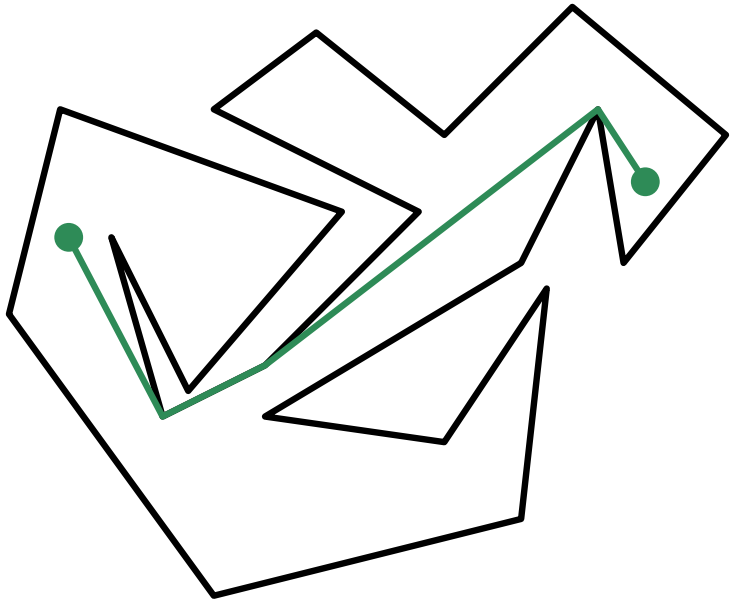

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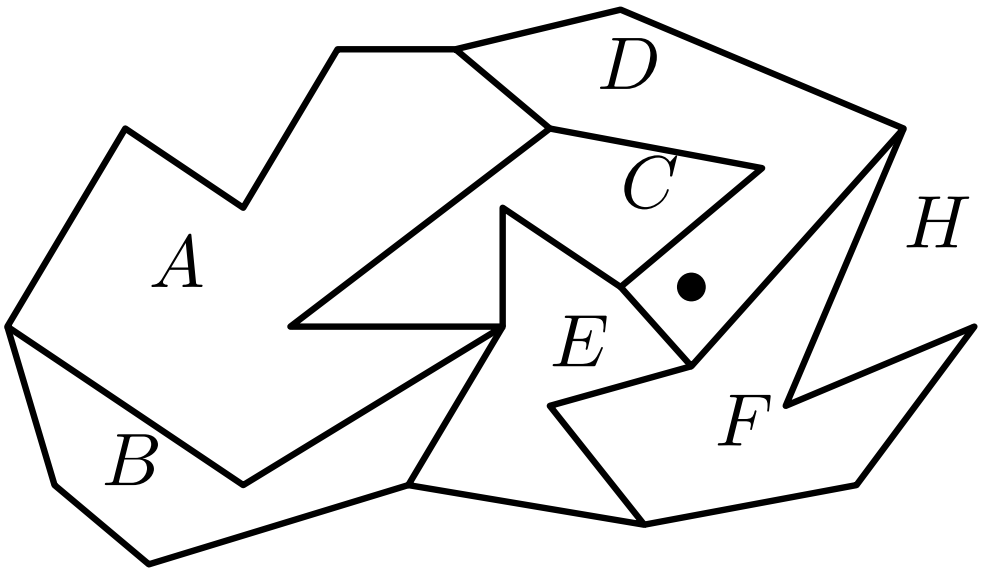

Use of Triangulations

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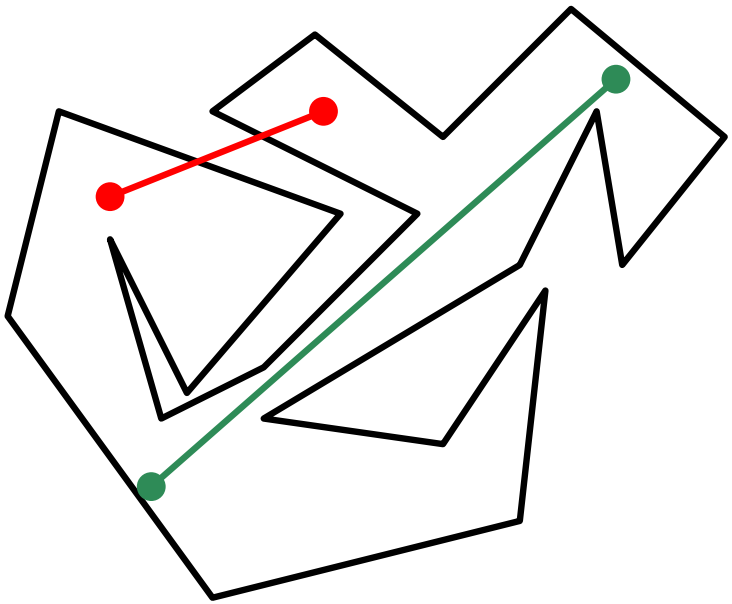

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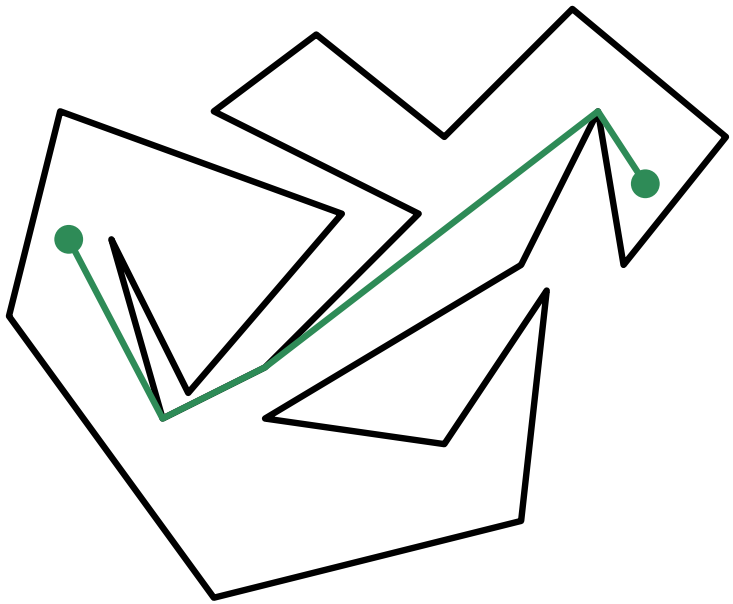

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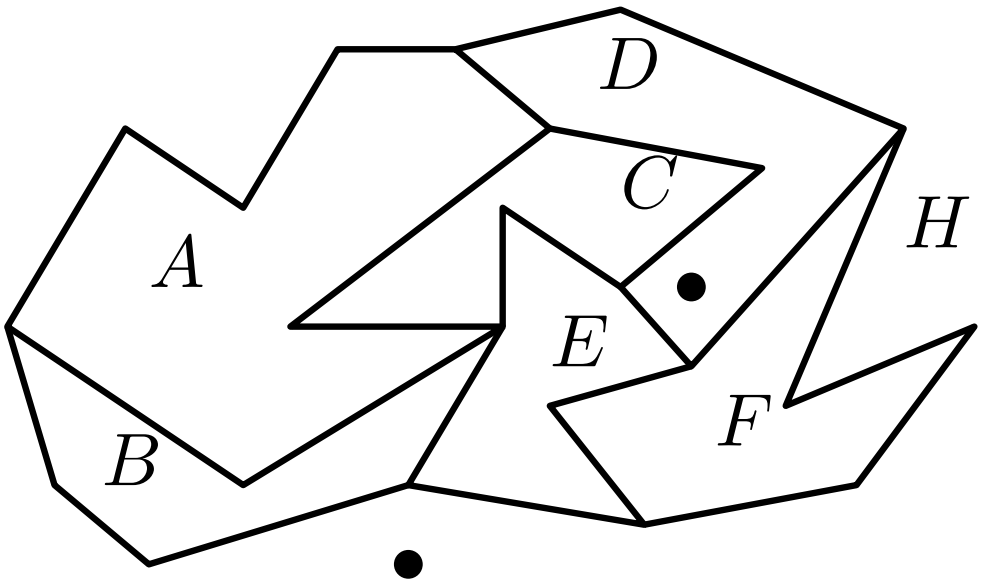

Use of Triangulations

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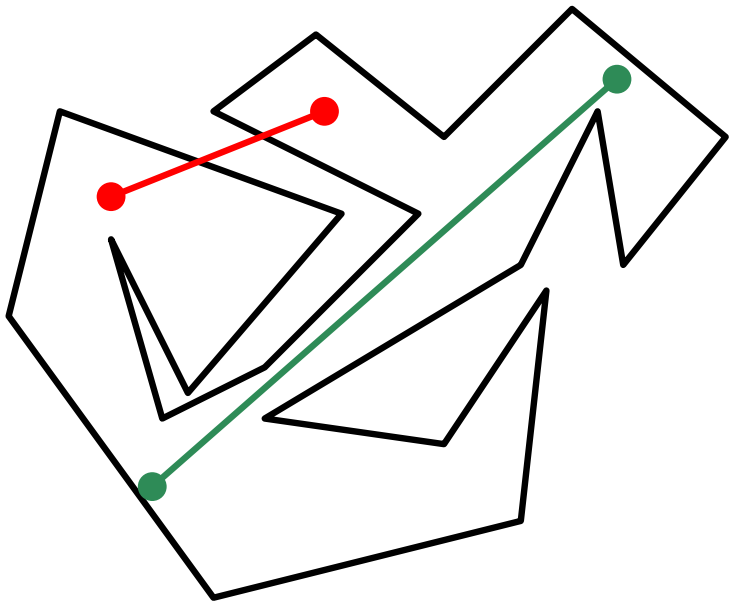

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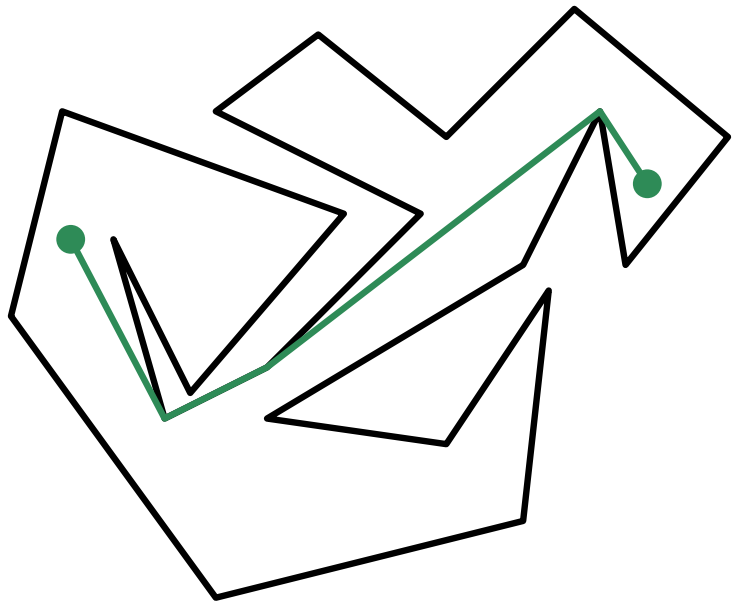

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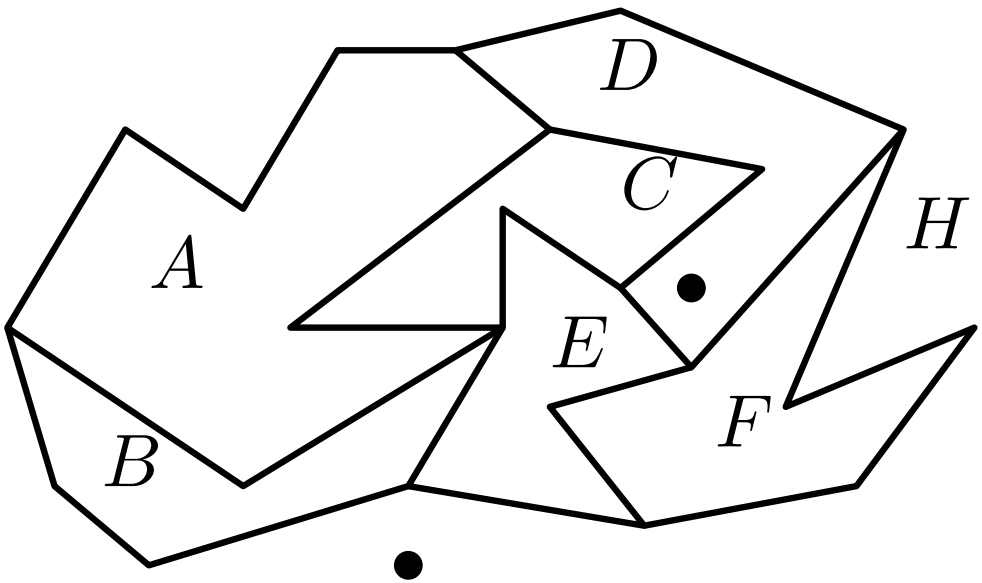
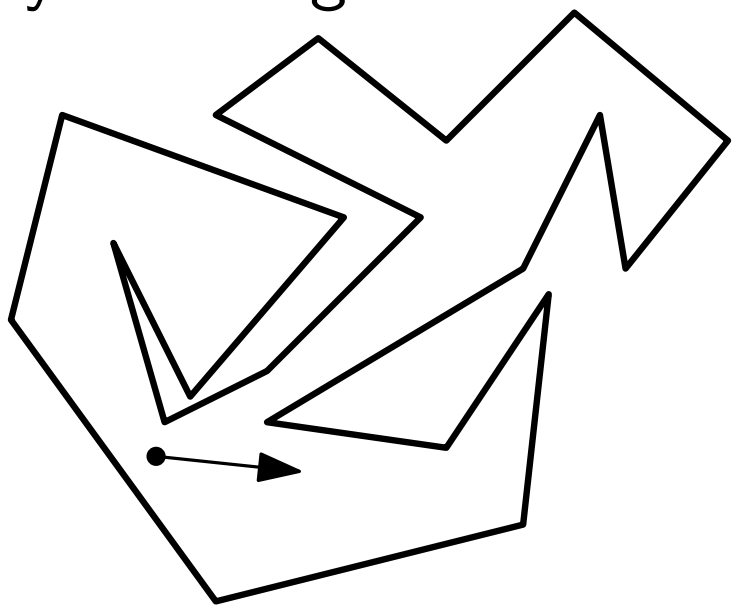
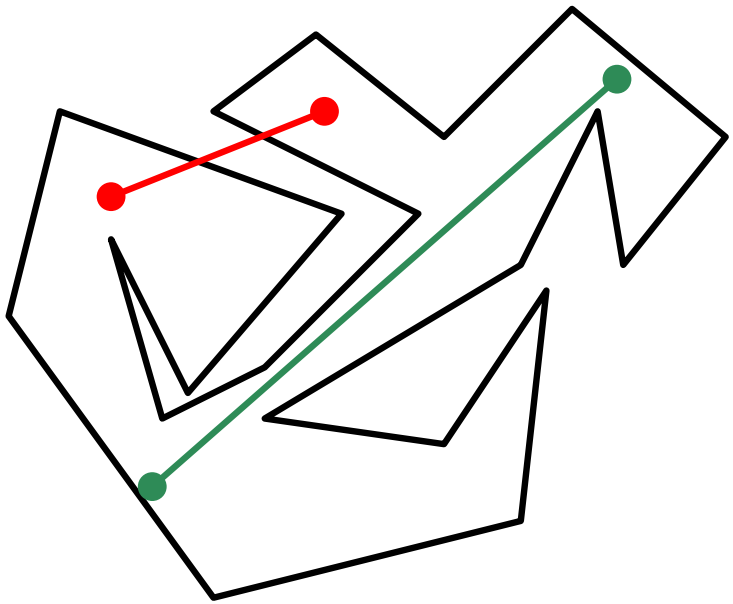


Ray shooting.

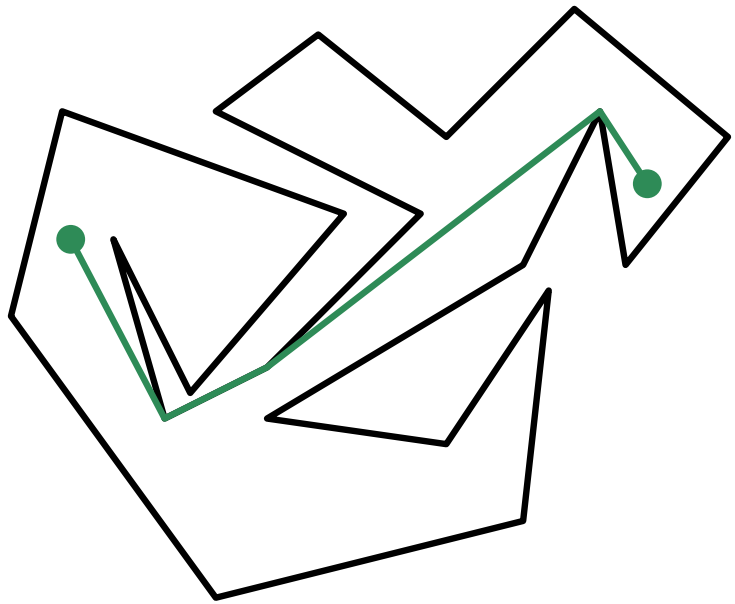


Use of Triangulations

Visibility problems.



Shortest paths.



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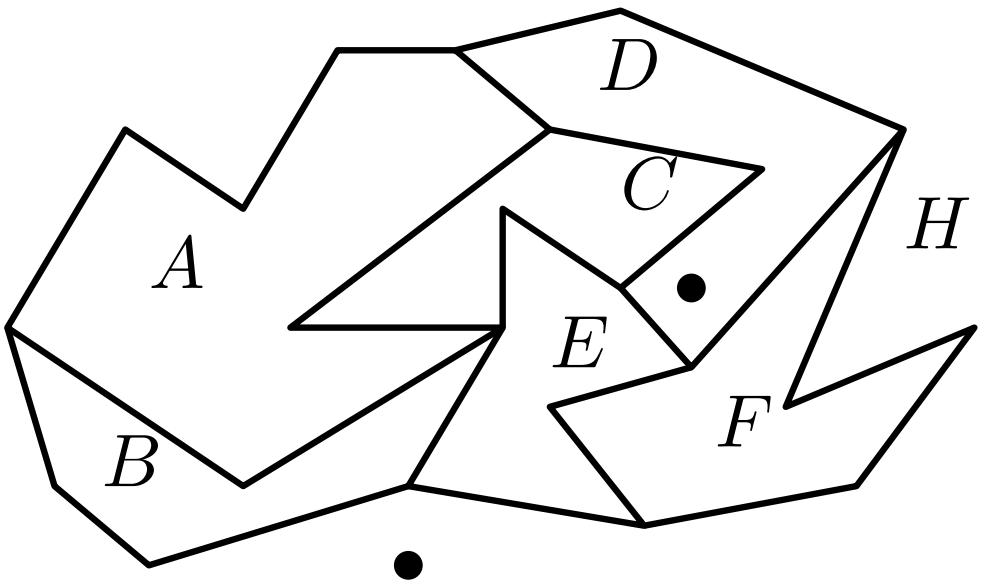

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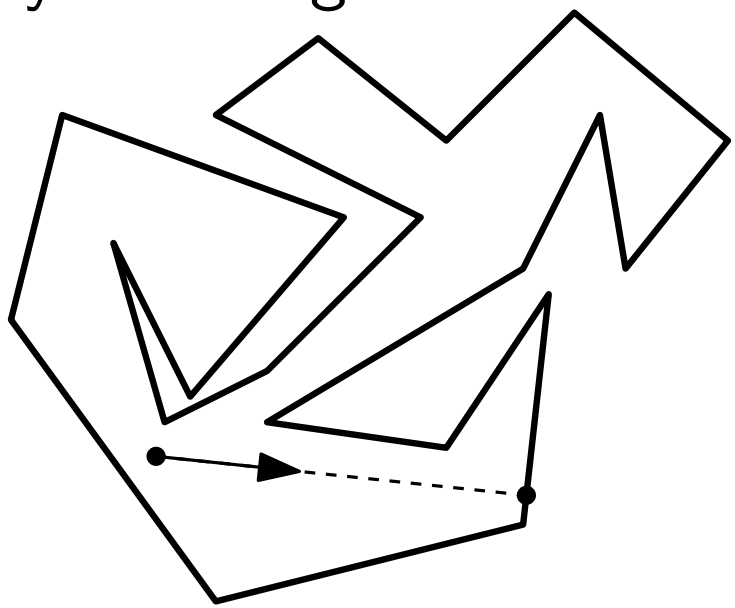
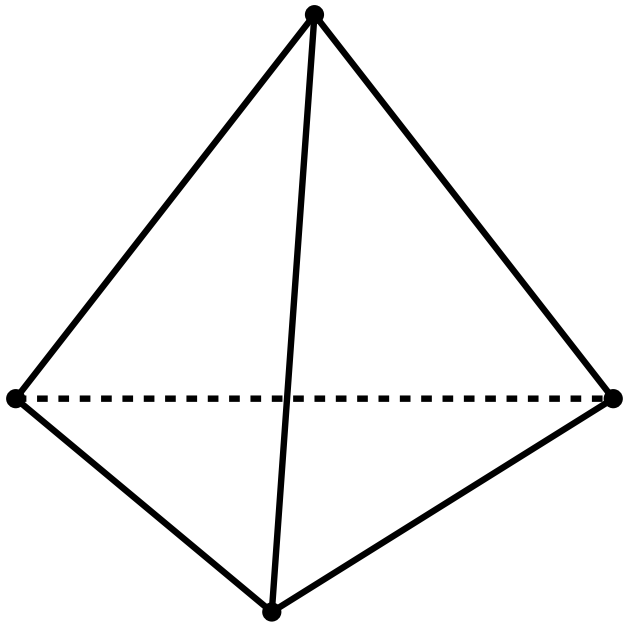
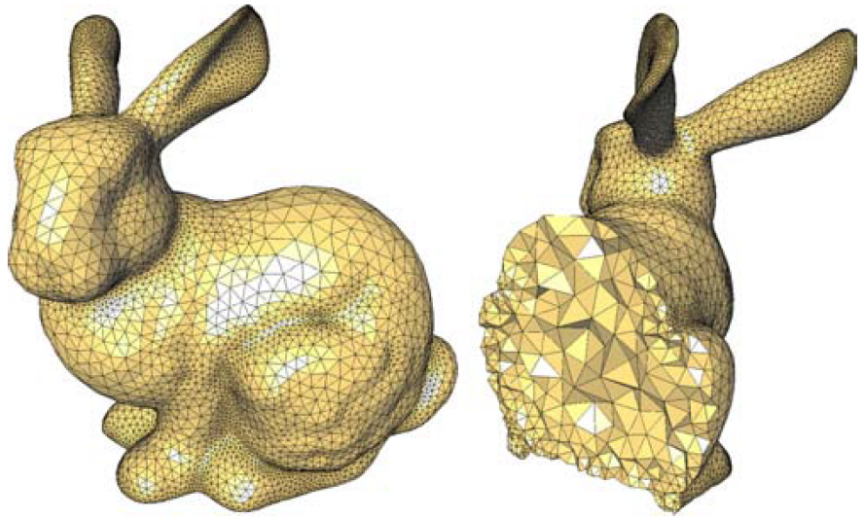


3D





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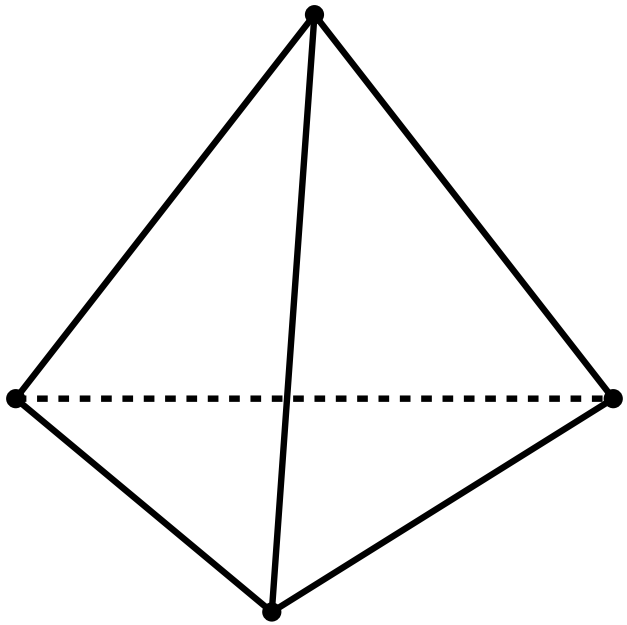

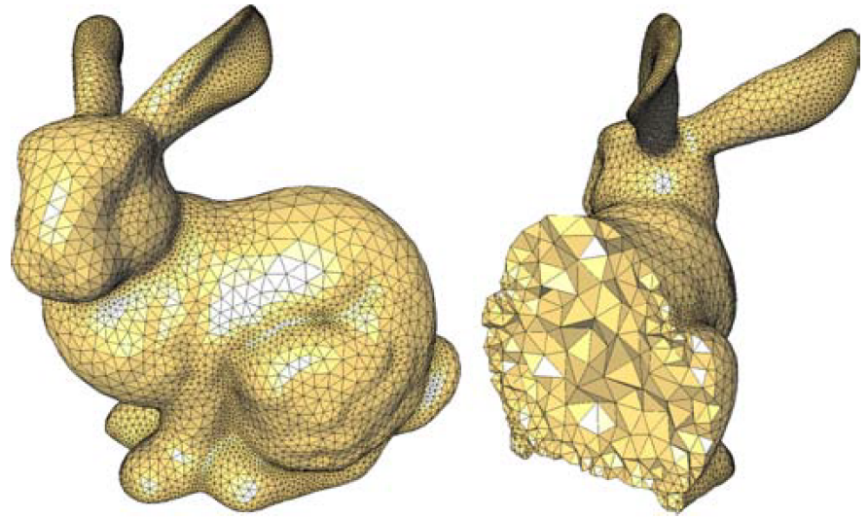

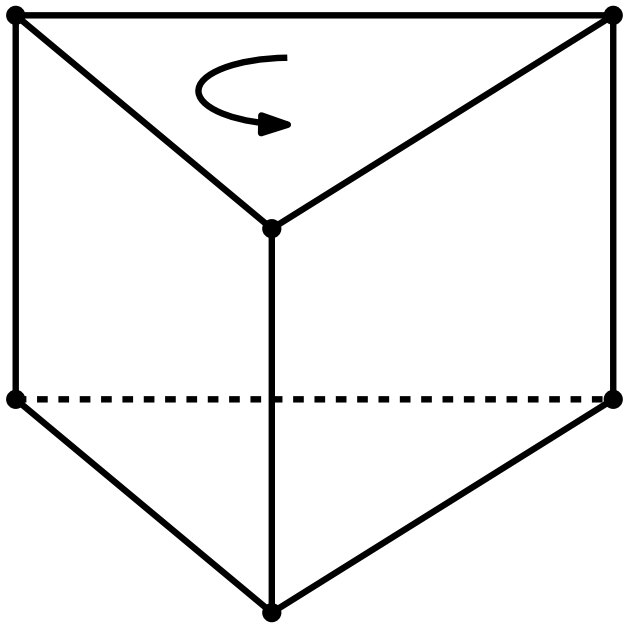

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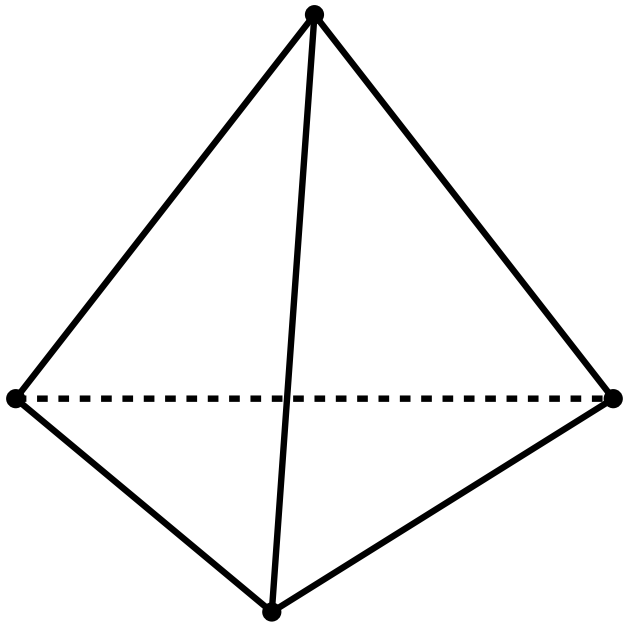

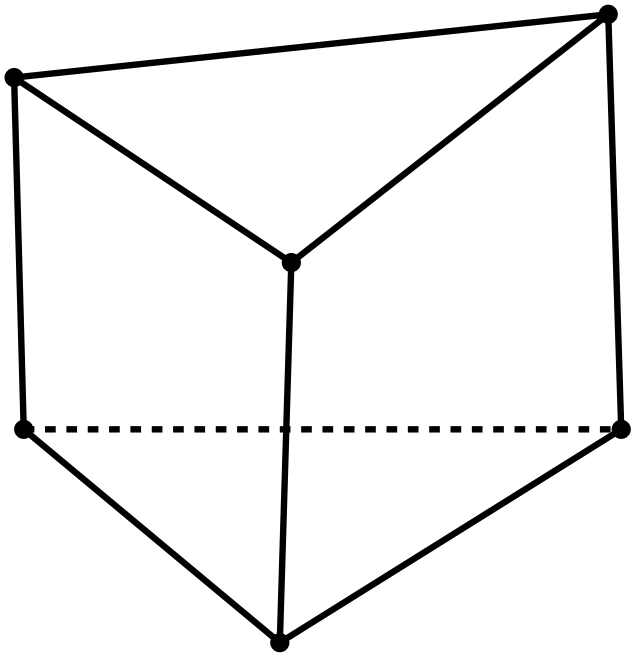

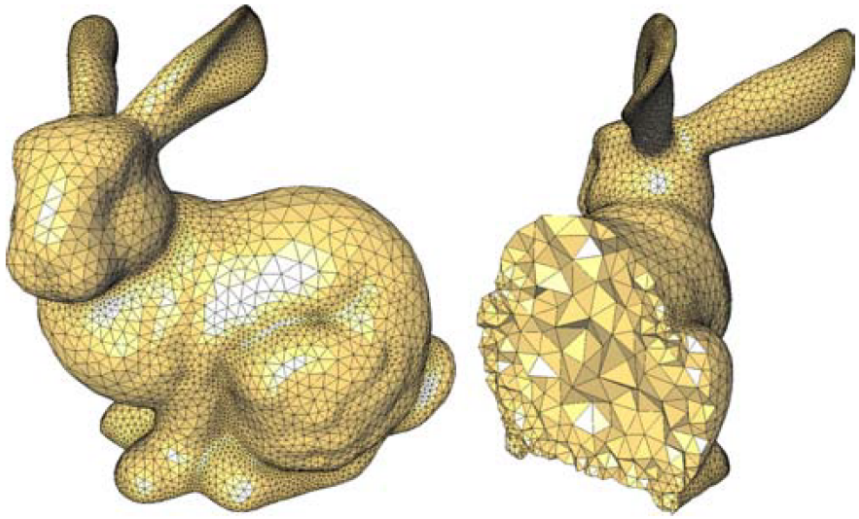

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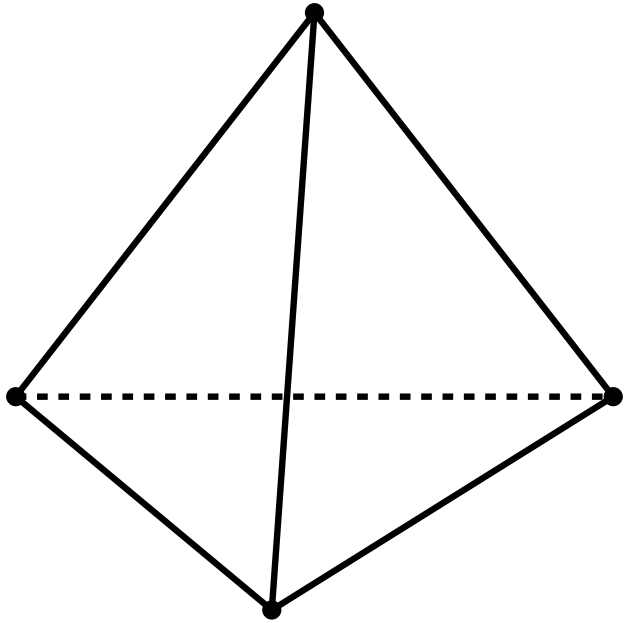

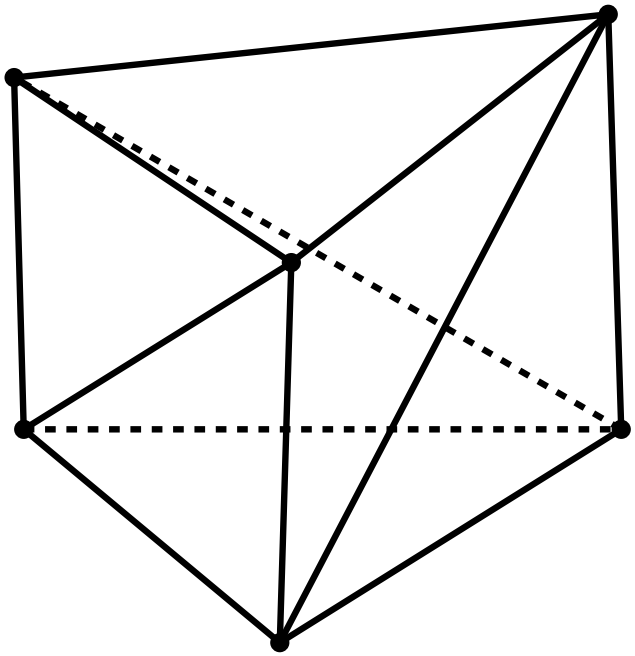

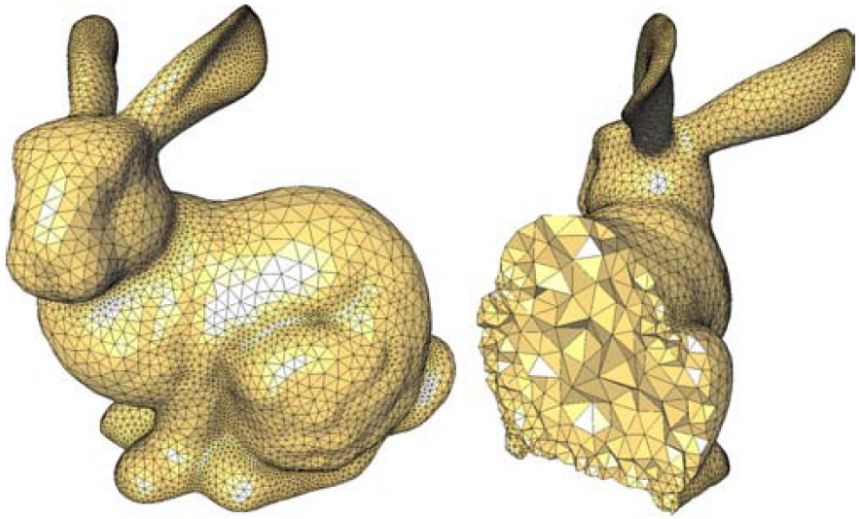

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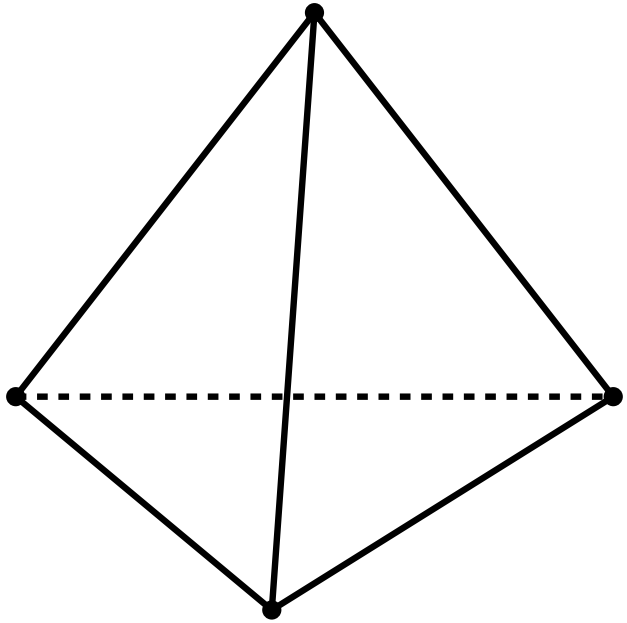

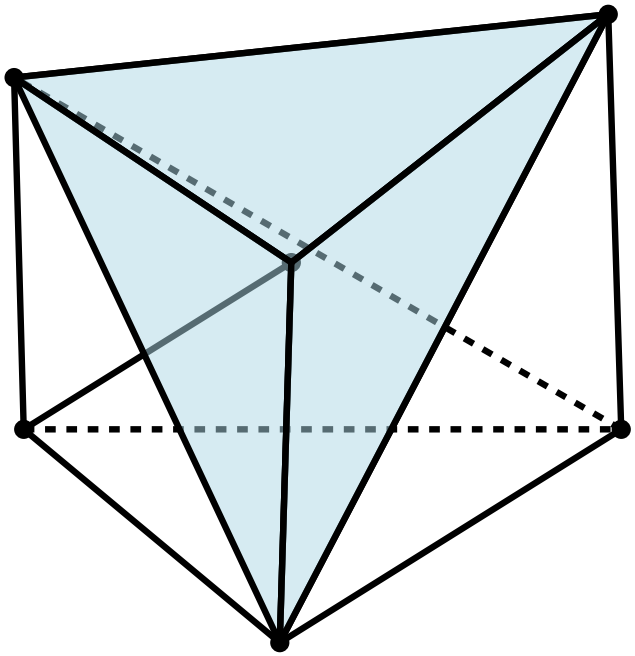

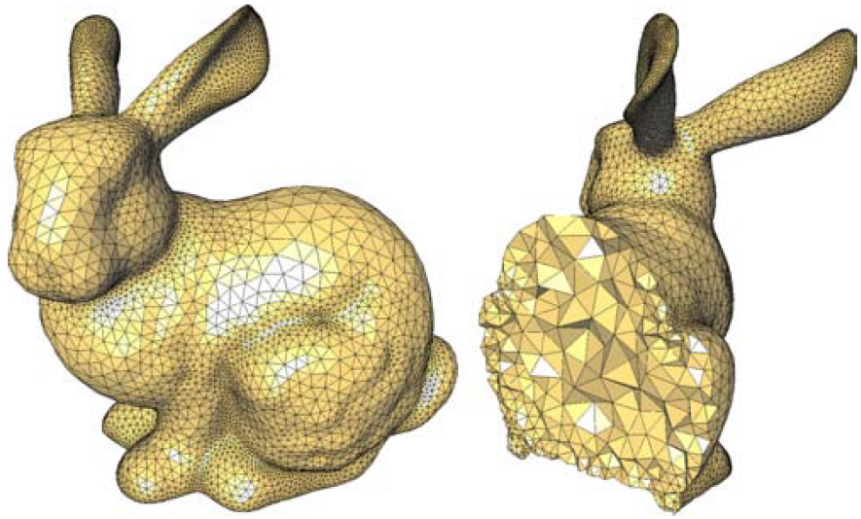

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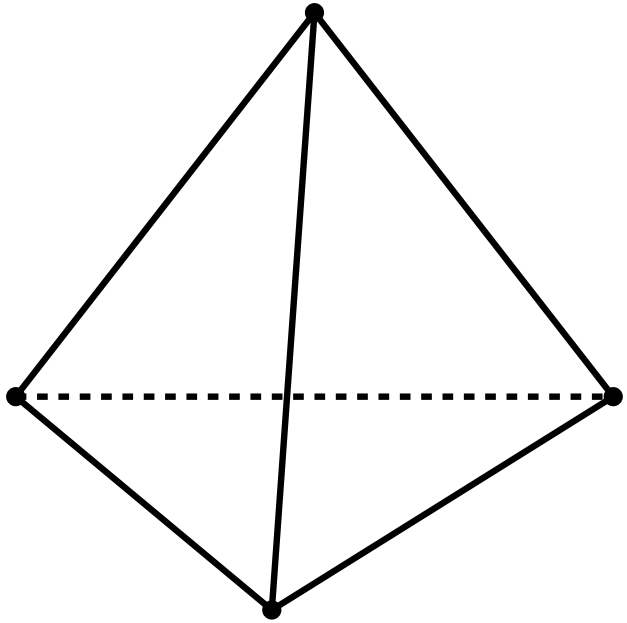

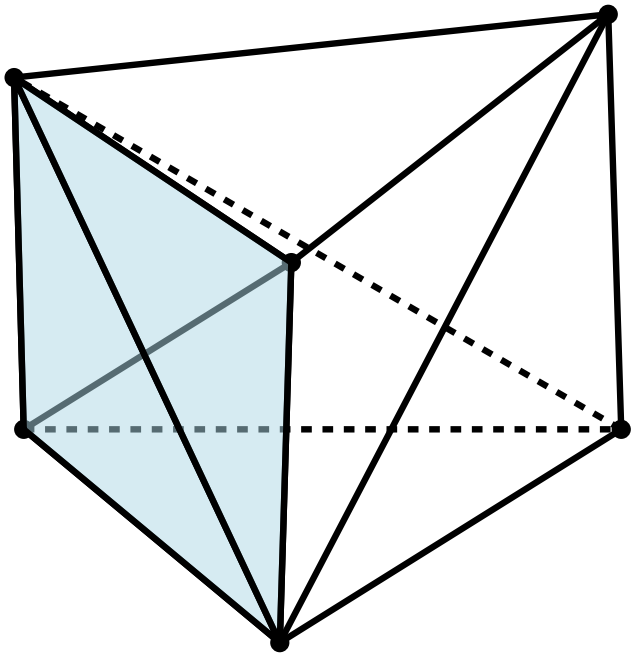

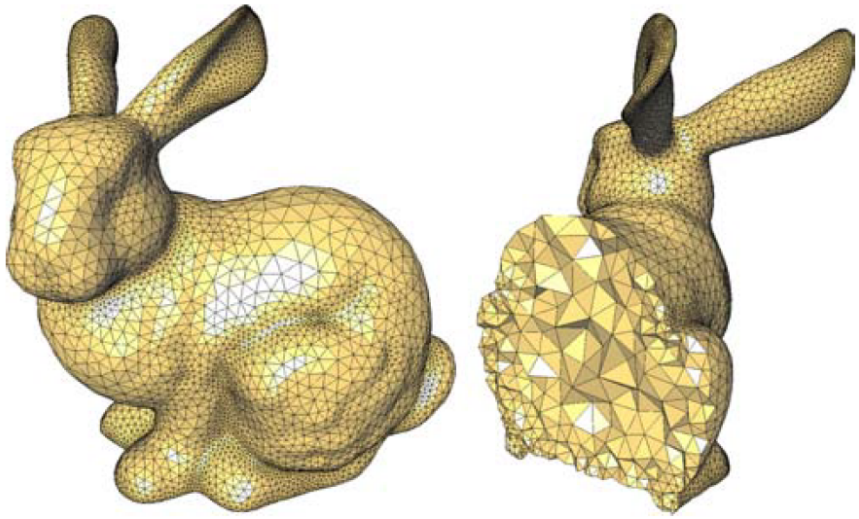

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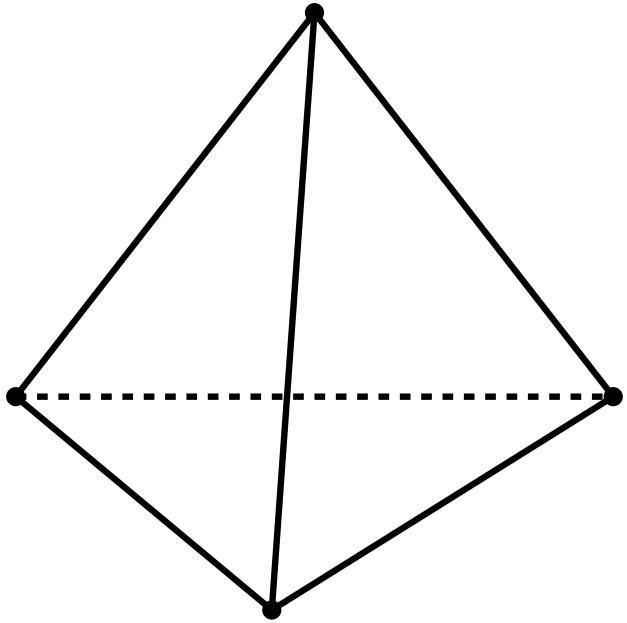

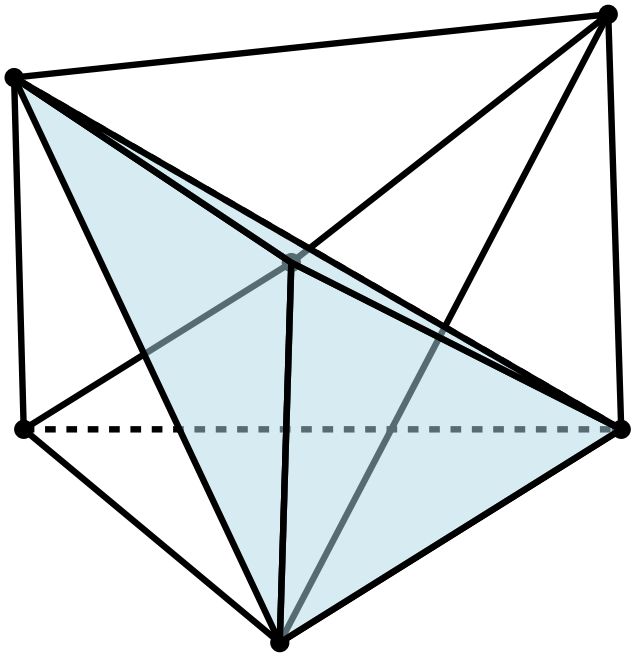

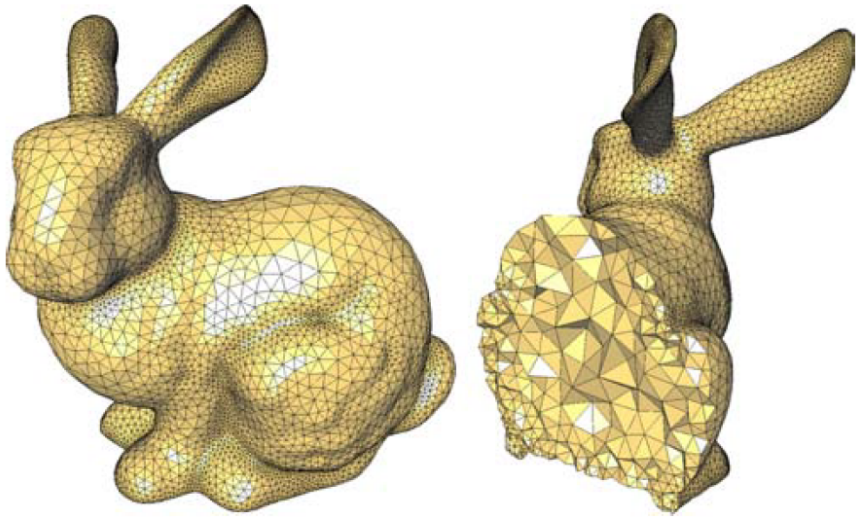

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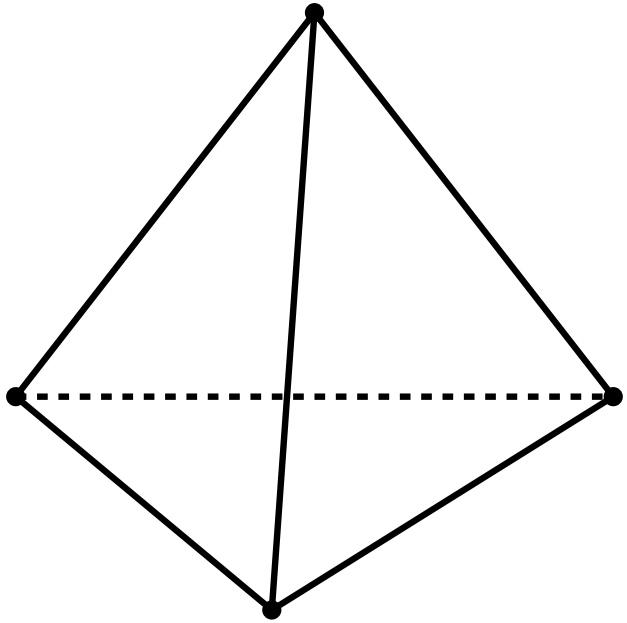
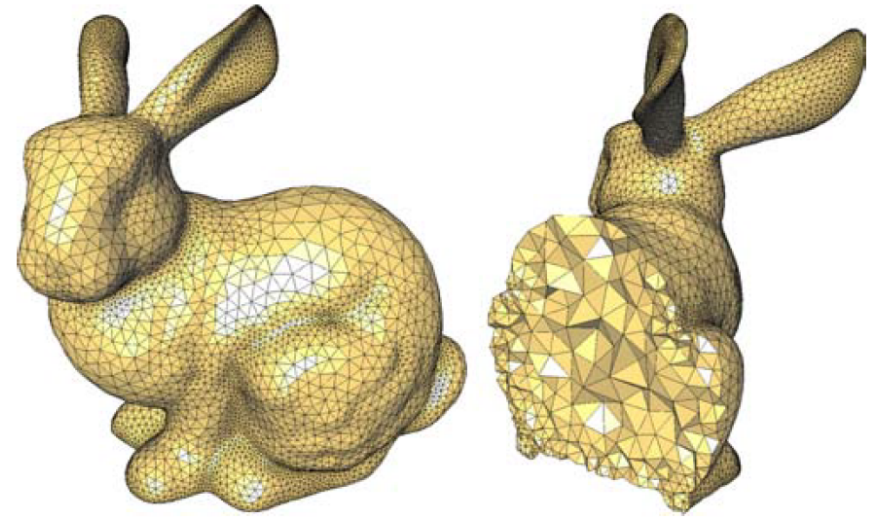
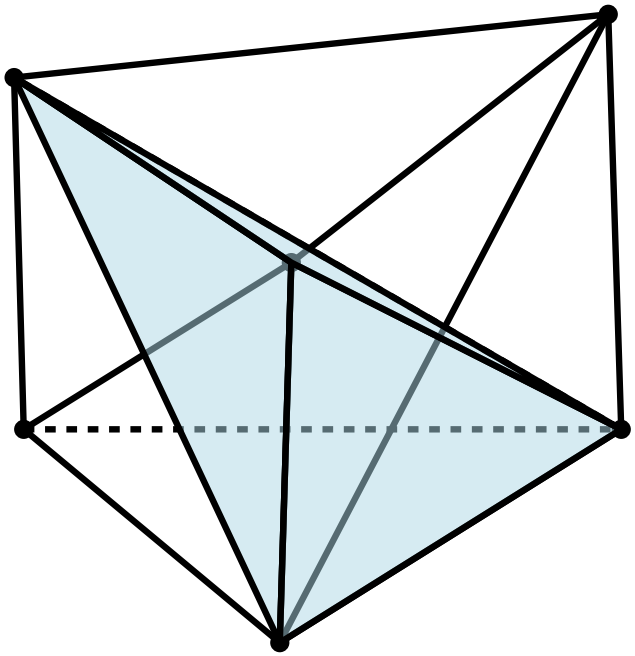






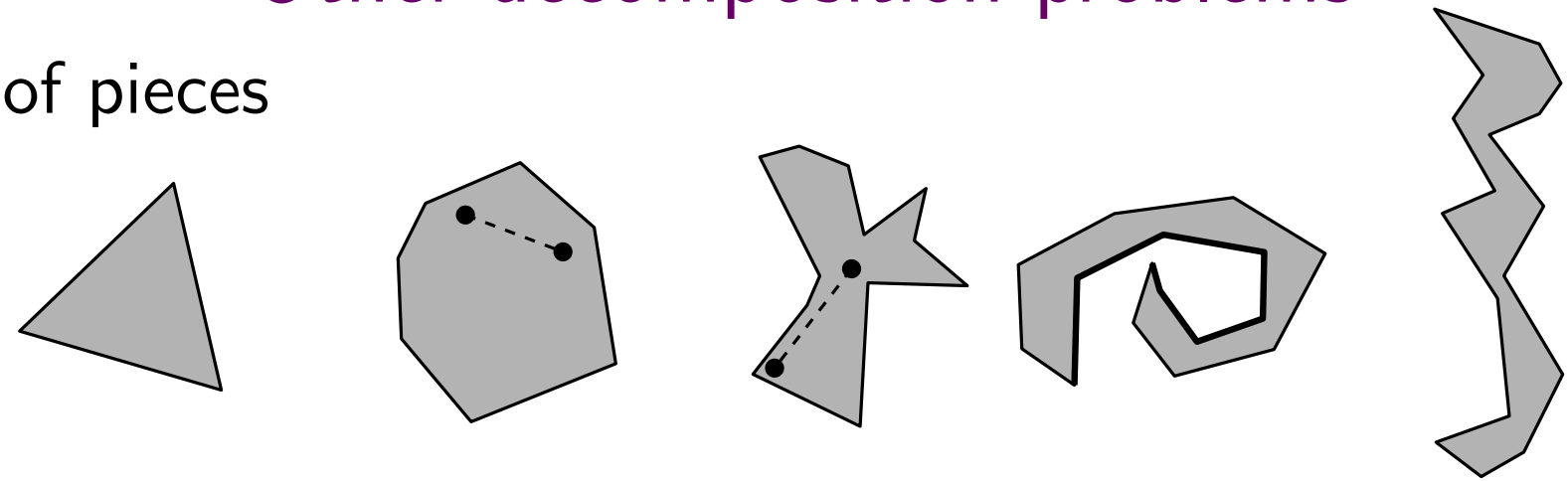
Sometimes $\Theta(n^2)$ Steiner points are needed. NP-complete to decide if possible without Steiner points.

Even when possible without Steiner points, it differs how many tetrahedra are needed.

For some convex polyhedra with n corners, there exist different tetrahedralizations without Steiner points of sizes $\Theta(n)$ and $\Theta(n^2)$.

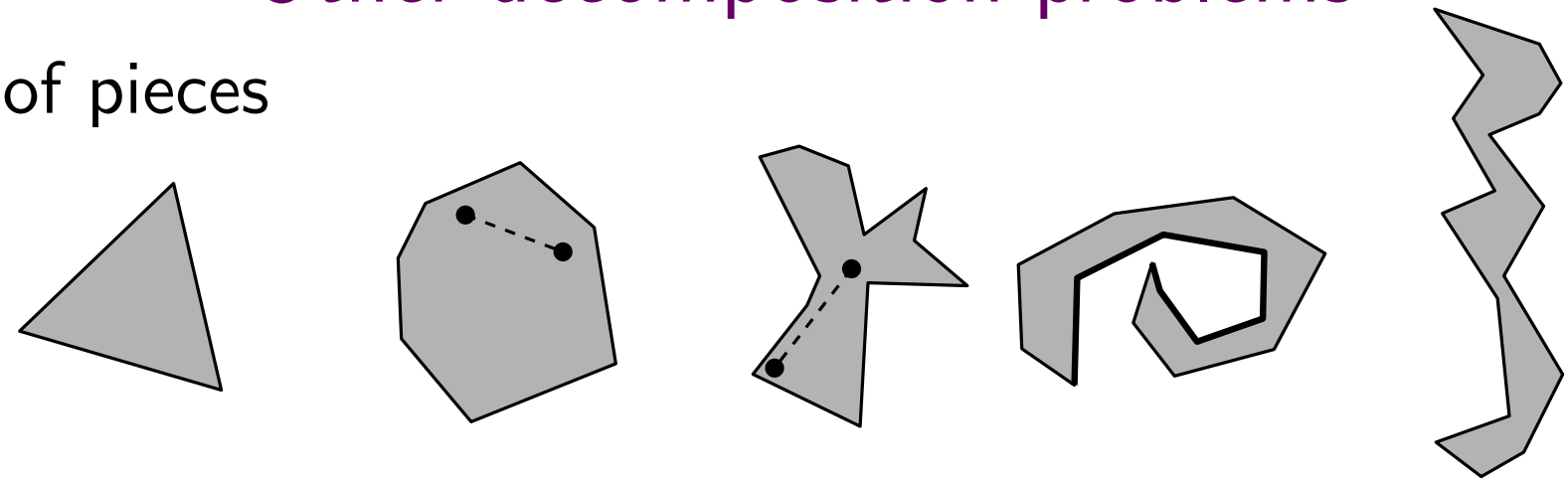
Other decomposition problems

Type of pieces

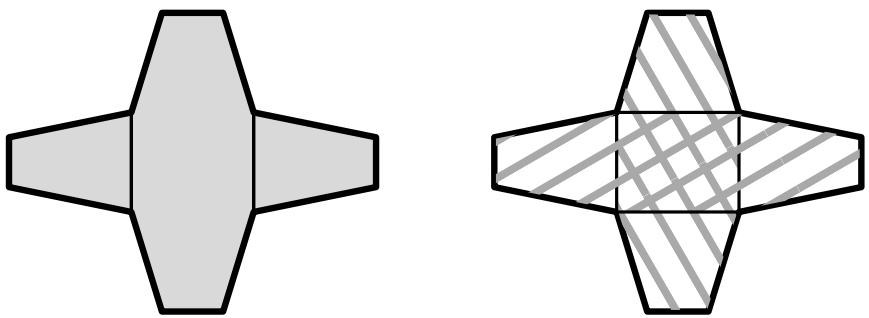


Other decomposition problems

Type of pieces

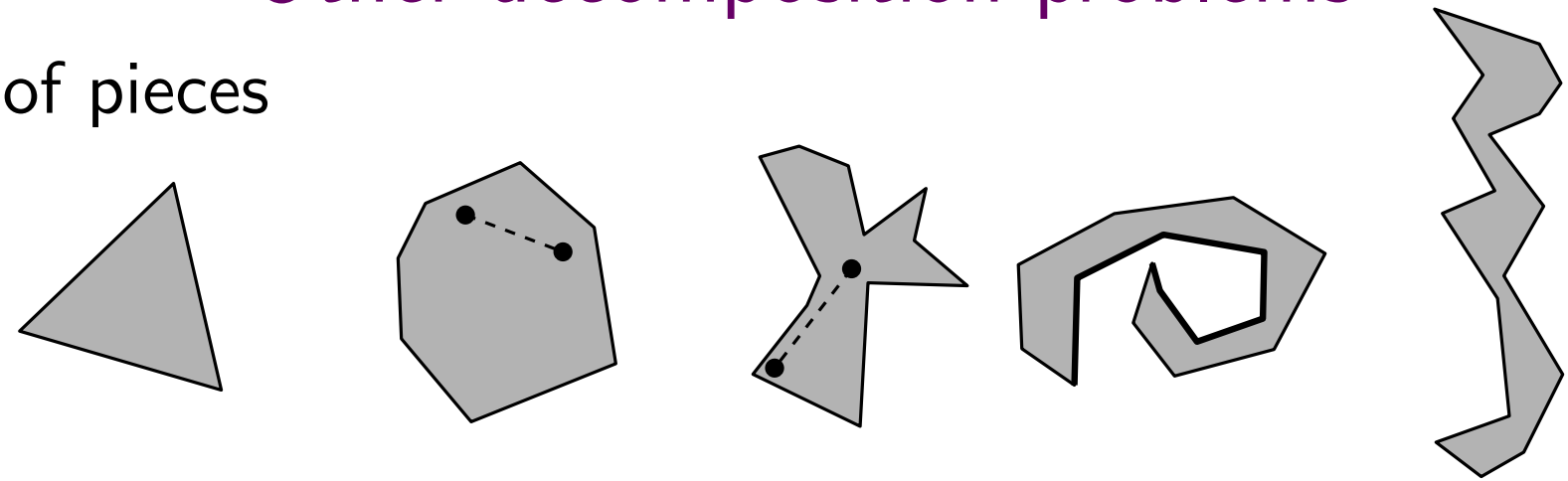


Partition vs. covering

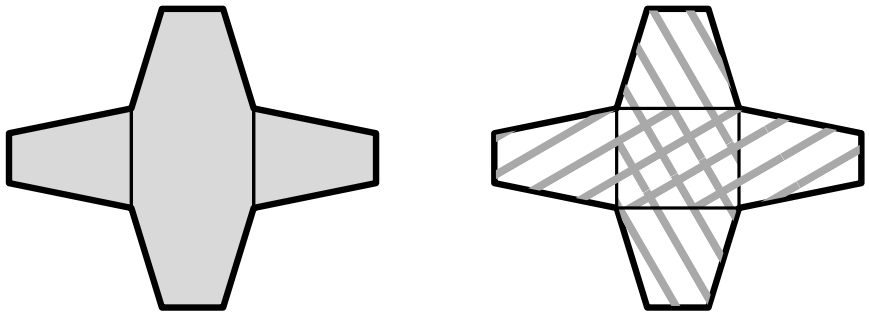


Other decomposition problems

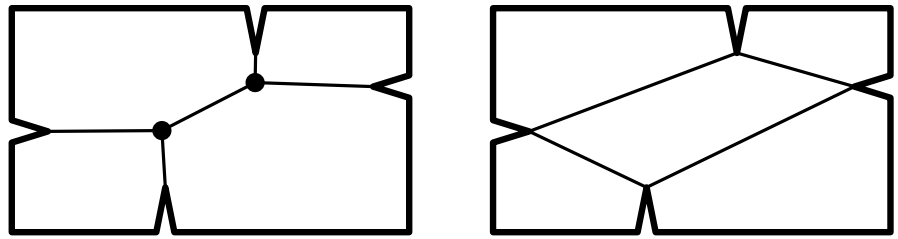
Type of pieces



Partition vs. covering

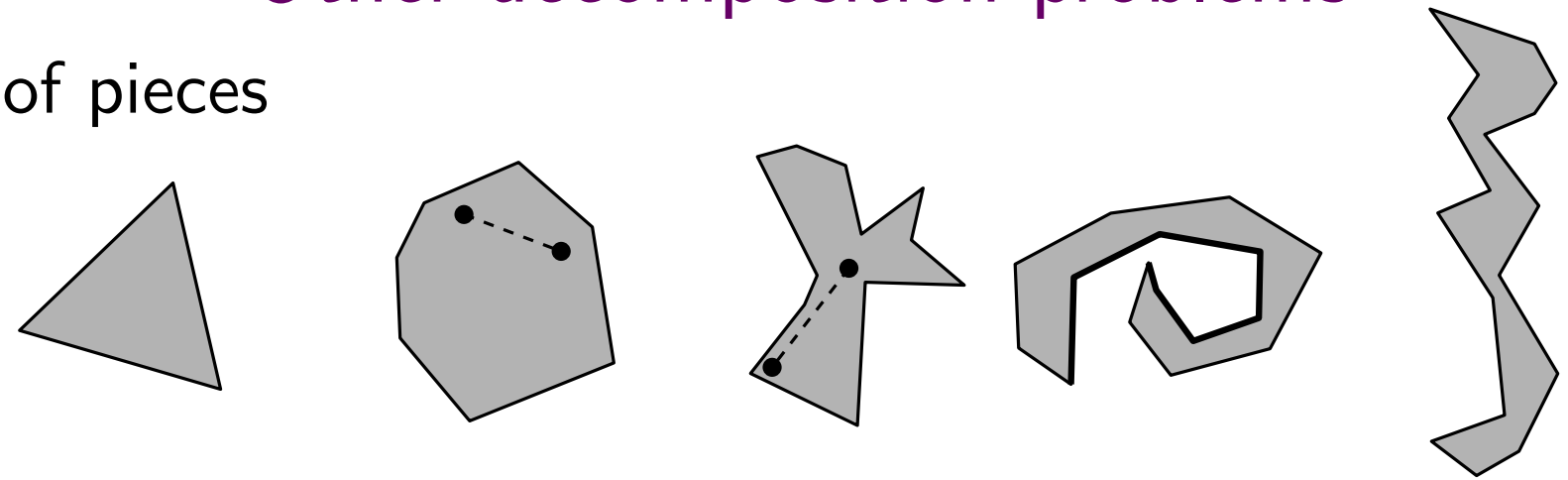


With or without Steiner points

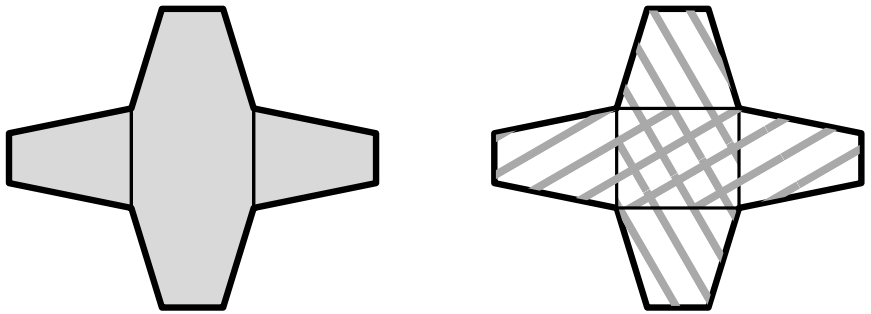


Other decomposition problems

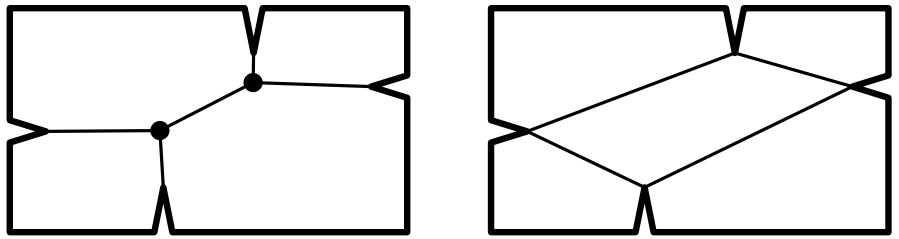
Type of pieces



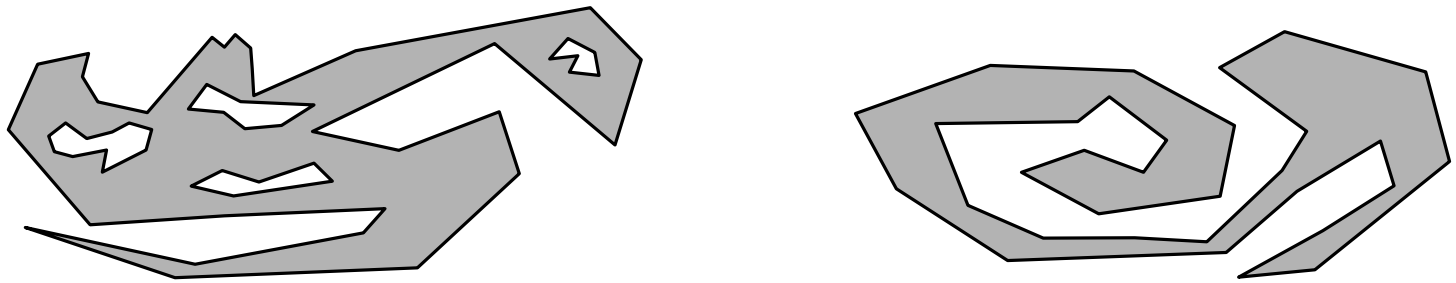
Partition vs. covering



With or without Steiner points

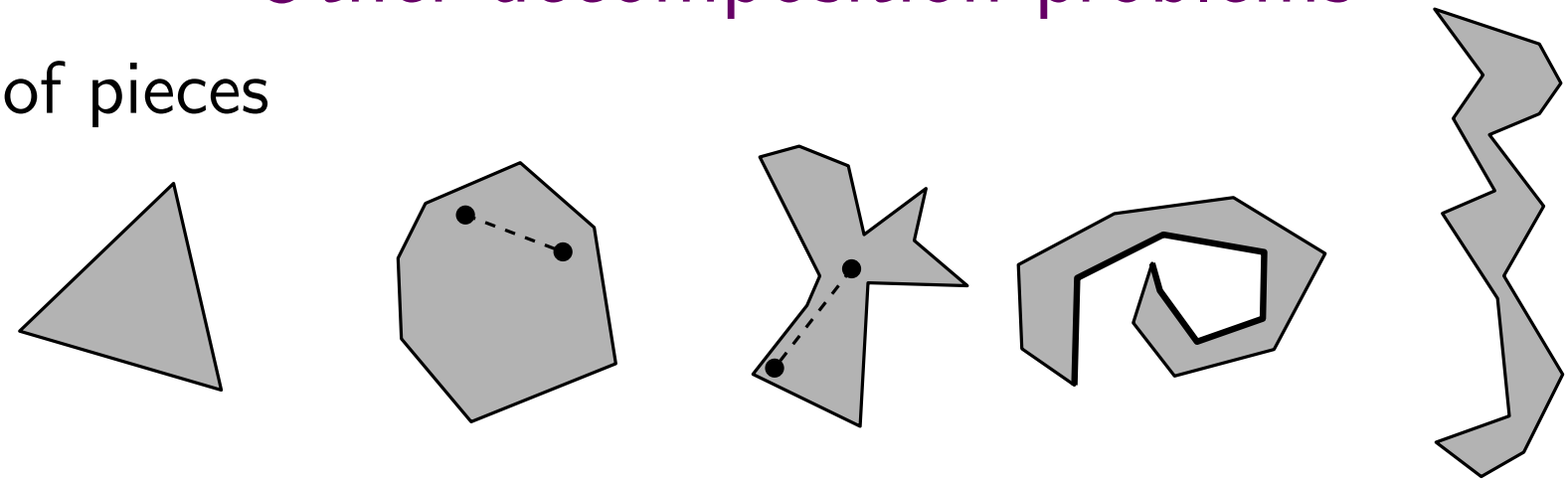


With or without holes

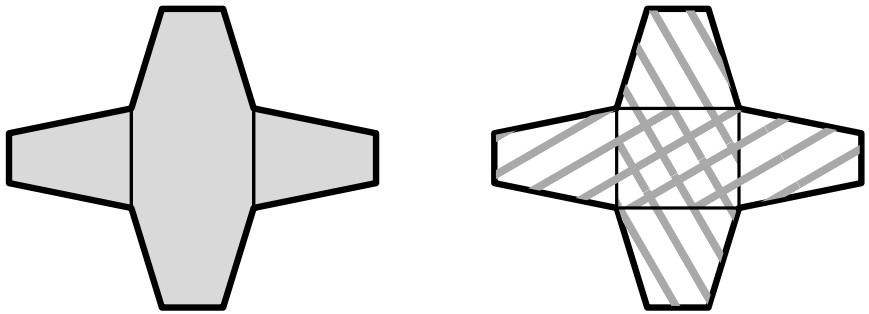


Other decomposition problems

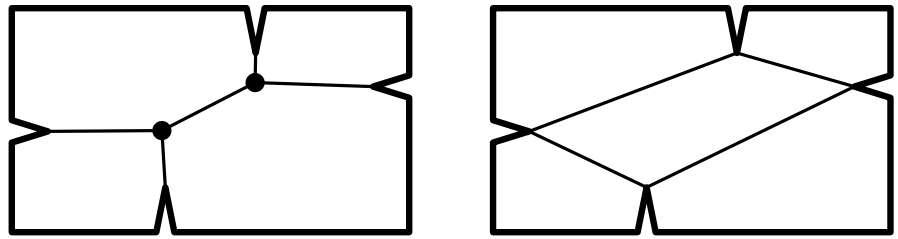
Type of pieces



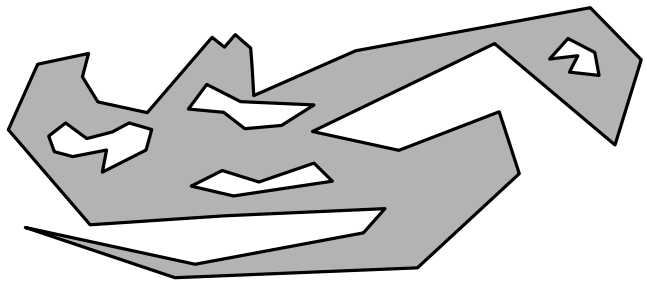
Partition vs. covering



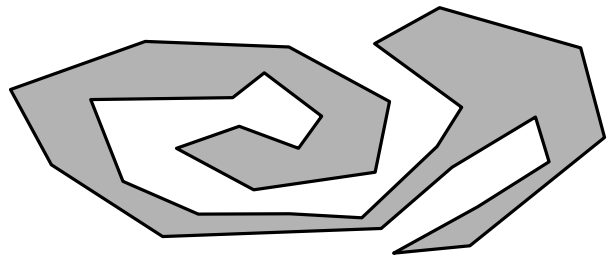
With or without Steiner points



With or without holes

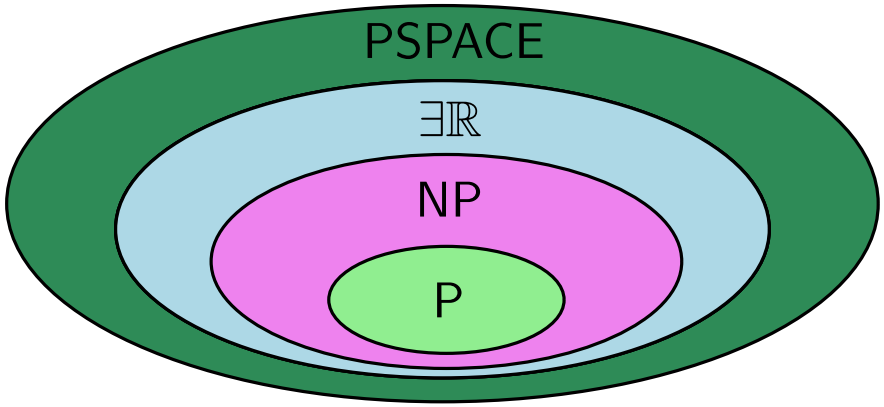


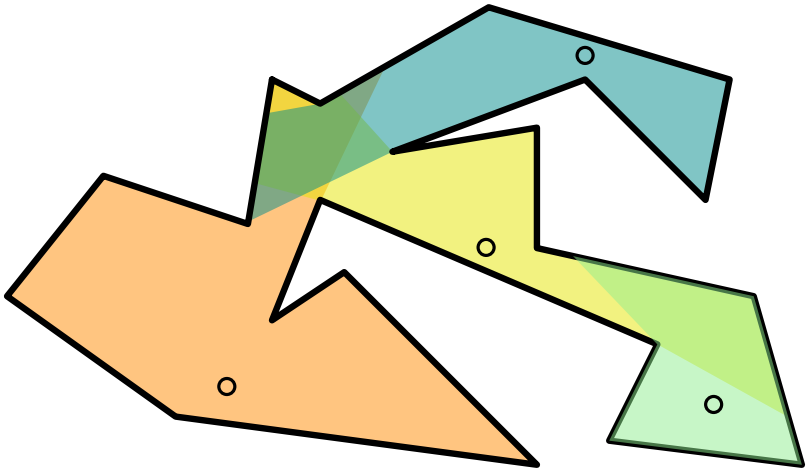
$5 \cdot 2 \cdot 2 \cdot 2 = 40$ problems!

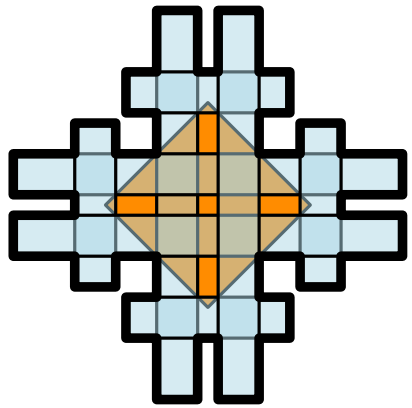


Some of my own work

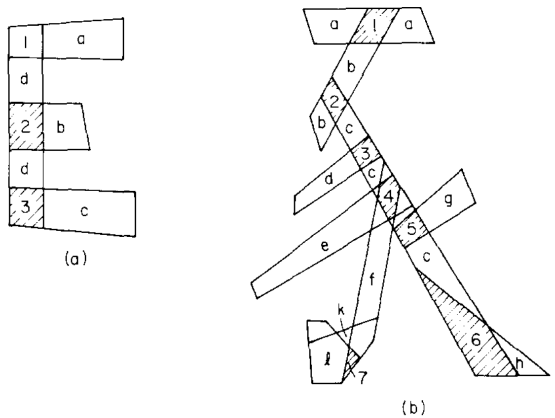
$\exists\mathbb{R}$: Given polynomial $P(x_1, \ldots, x_k)$ with integer coefficients, exists real solution to $P(x_1, \ldots, x_k) = 0$?







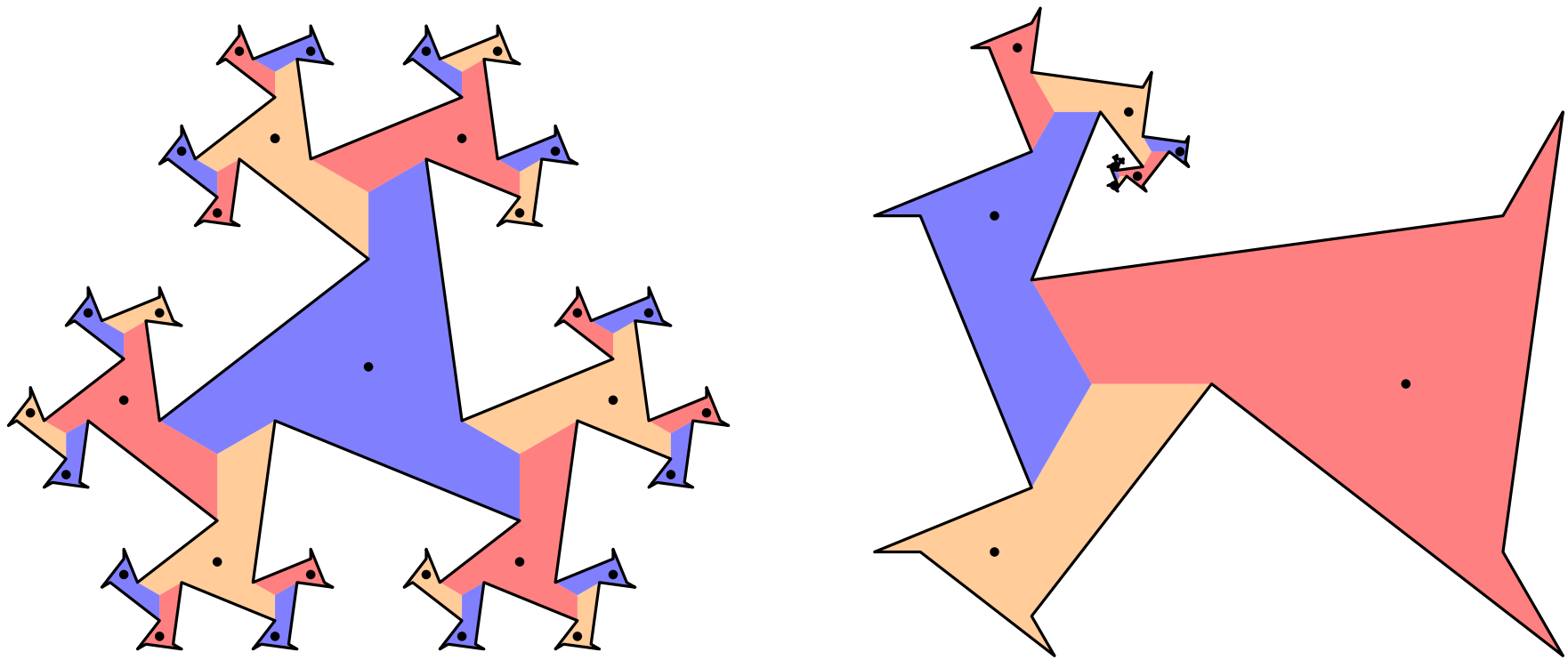
Art Gallery Problem is $\exists\mathbb{R}$-complete (**A.**, Adamaszek, Miltzow, 2018)



Minimum Convex Cover is $\exists\mathbb{R}$-complete (**A.**, 2021)

Recent work on decomposition problems

Minimum star partitions in polynomial time ($O(n^{107})$)

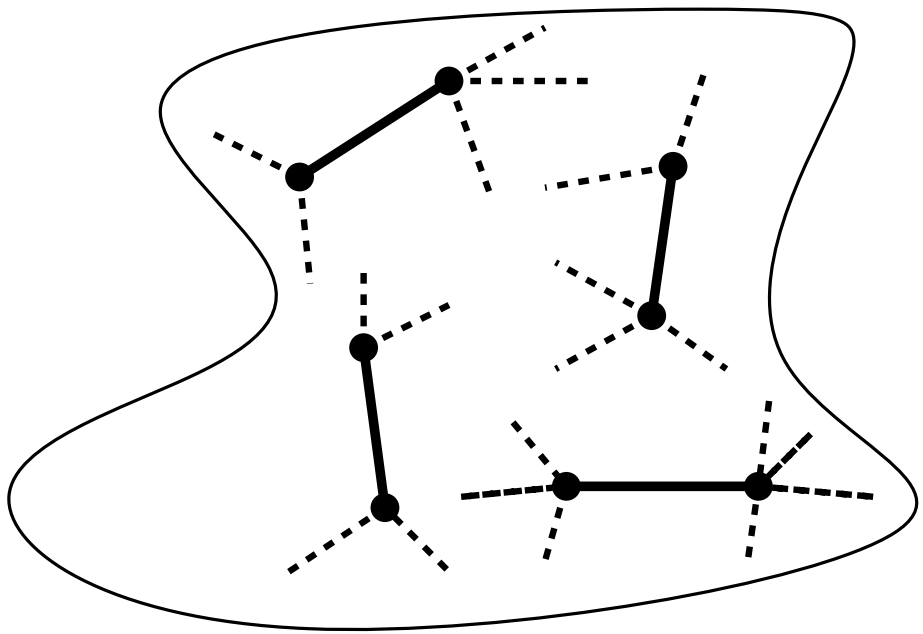


A., Blikstad, Nusser, Zhang, 2023

Introduction to Approximation Algorithms, part I

08-1 2025, Srikanth Srinivasan,
DIKU (Slides: Mikkel Abrahamsen)

APPROX-VERTEX-COVER(G)
  $C := \emptyset$
  while $E(G) \neq \emptyset$
    choose $uv \in E(G)$
    $C := C \cup \{u, v\}$
    remove all edges incident on u or v from $E(G)$
  return C



The big picture

Previously: Fast exponential algorithms (good for small instances) and parameterized algorithms (good for special cases).

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Today: Approximation algorithms (good when suboptimal solutions are acceptable).

Definition

Def.: An algorithm for an optimization problem has *approximation ratio* $\rho(n)$ if for every input of size n,

$$\max\left\{\frac{C}{C^*}, \frac{C^*}{C}\right\} \leq \rho(n).$$

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minimization problem

maximization problem

Vertex Cover

Def.: Let $G = (V, E)$ be a graph. A set $V' \subseteq V$ of vertices is a *vertex cover* if for all $uv \in E$, we have $u \in V'$ or $v \in V'$.

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NP-hard!

```
APPROX-VERTEX-COVER(G)
  C := ∅
  while E(G) ≠ ∅
    choose uv ∈ E(G)
    C := C ∪ {u, v}
    remove all edges incident on u or v from E(G)
  return C
```

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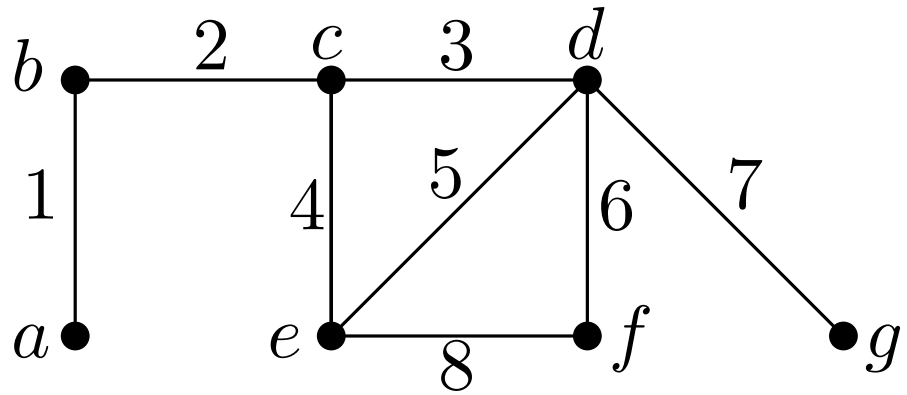
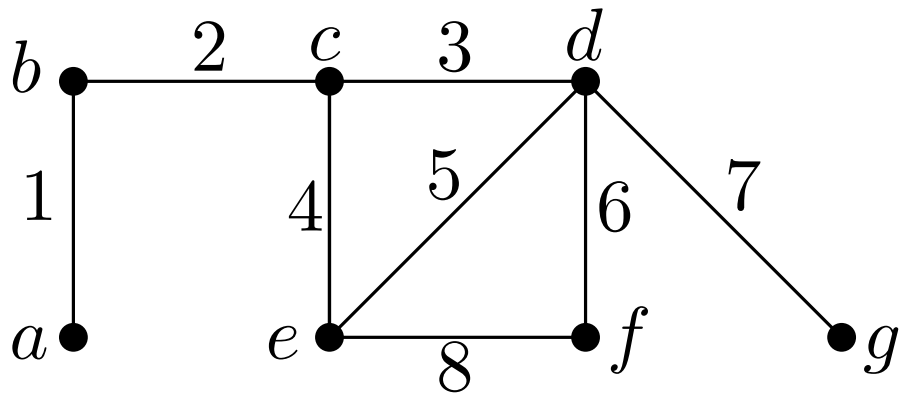
Exercise:



Implementation



Adjacency lists:

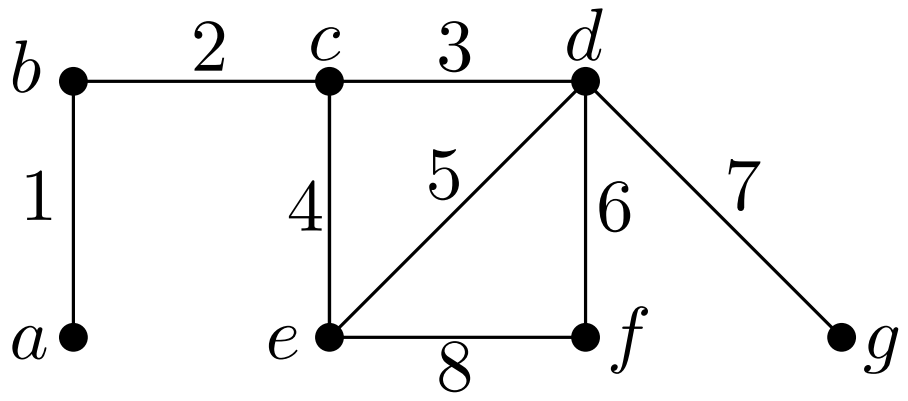
$L[a] = \{b\}$

$L[b] = \{a, c\}$

$L[c] = \{b, d, e\}$

$\vdots$

Implementation



Adjacency lists:

$L[a] = \{b\}$ $E[a] = [1]$

$L[b] = \{a, c\}$ $E[b] = [1, 2]$

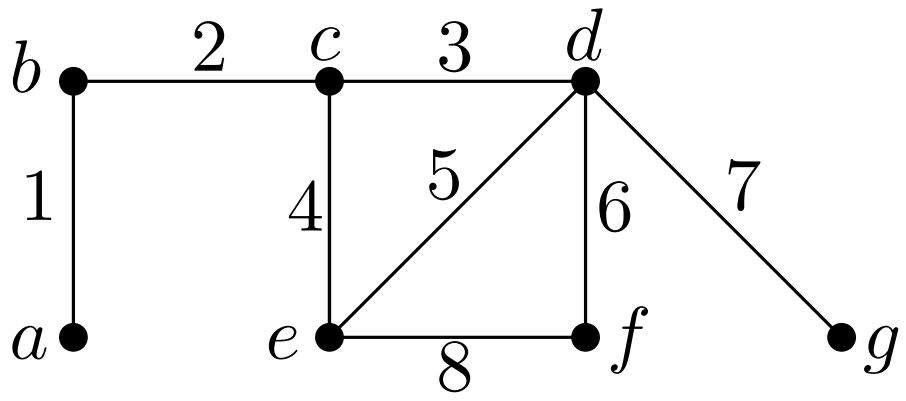
$L[c] = \{b, d, e\}$ $E[c] = [2, 3, 4]$

$\vdots$ $\vdots$

Array of edges

$[(a, b, 1),\ (b, c, 1),\ (c, d, 1),\ (c, e, 1), (d, e, 1),\ \ldots]$

Implementation



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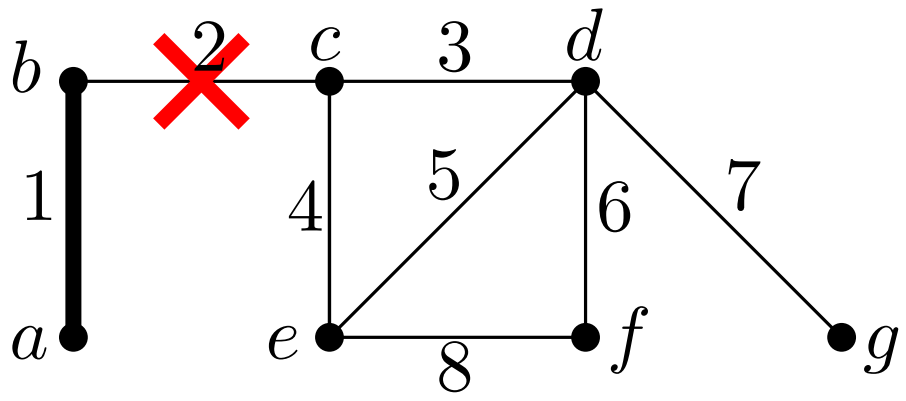

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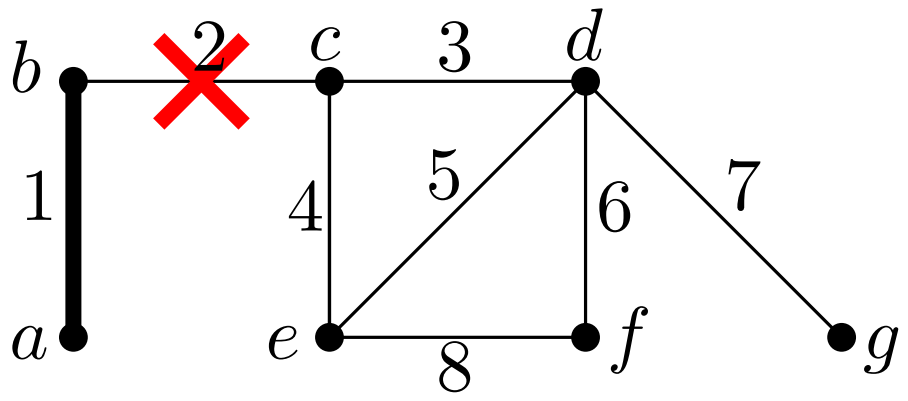
Array of edges

$[(a, b, 1),\ (b, c, 1),\ (c, d, 1),\ (c, e, 1), (d, e, 1),\ \ldots]$

$[(a, b, 0),\ (b, c, 0),\ (c, d, 1),\ (c, e, 1), (d, e, 1),\ \ldots]$

Implementation



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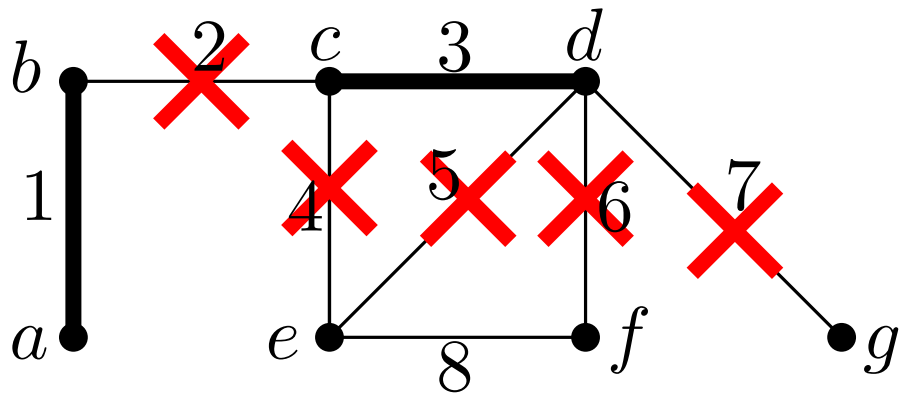

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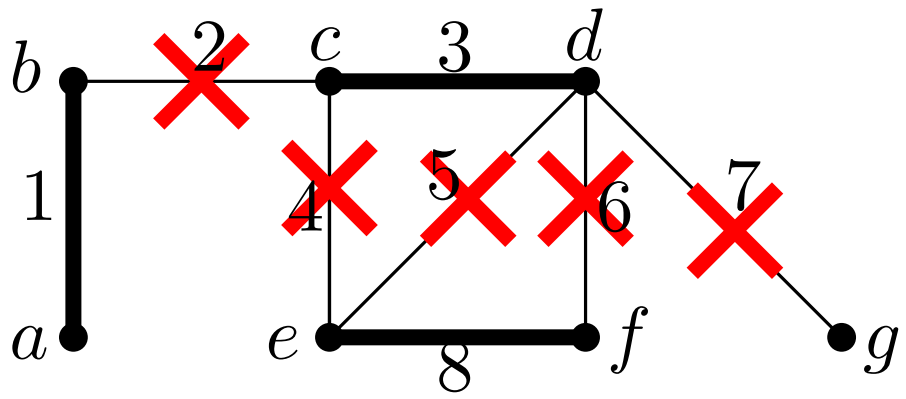

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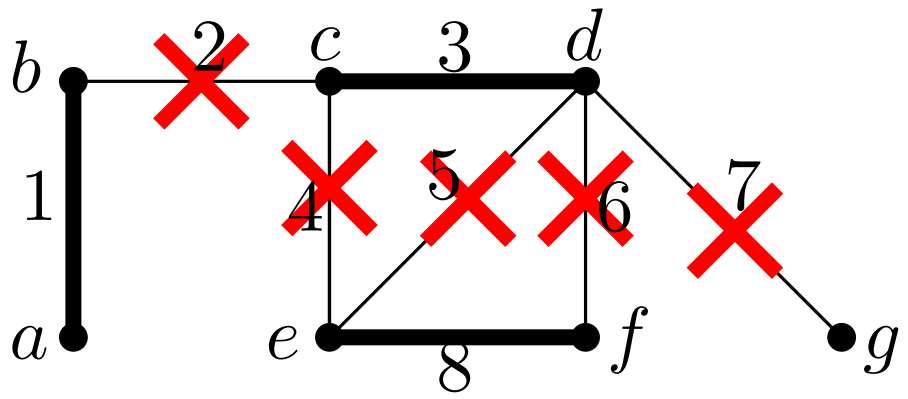

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Running time: $O(|V| + |E|)$

Theorem

Thm.: APPROX-VERTEX-COVER is a 2-approximation algorithm.

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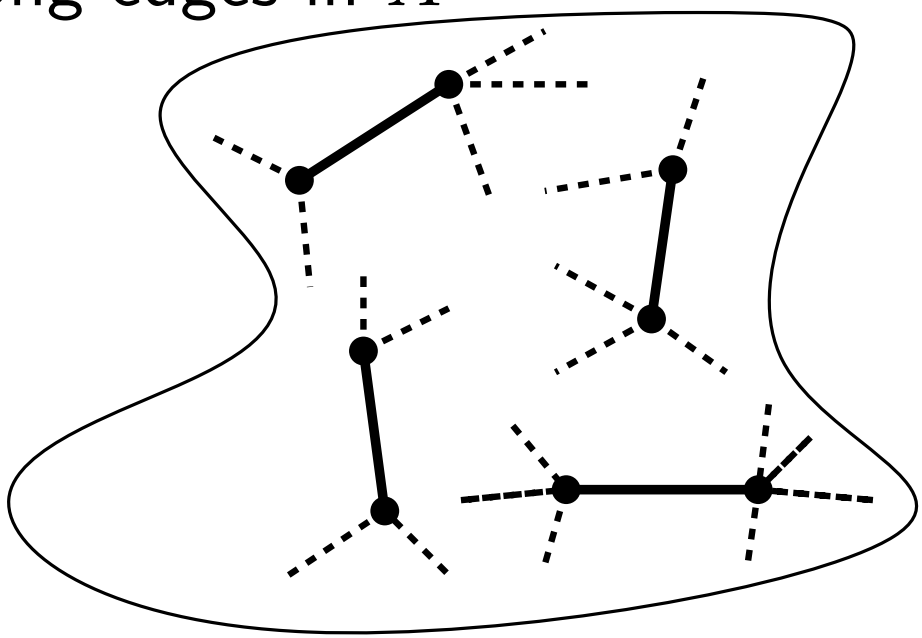
Proof: Let C^* be an optimal cover.

Let $A \subset E$ be the edges chosen by the algorithm.

Obs: No shared endpoints among edges in A



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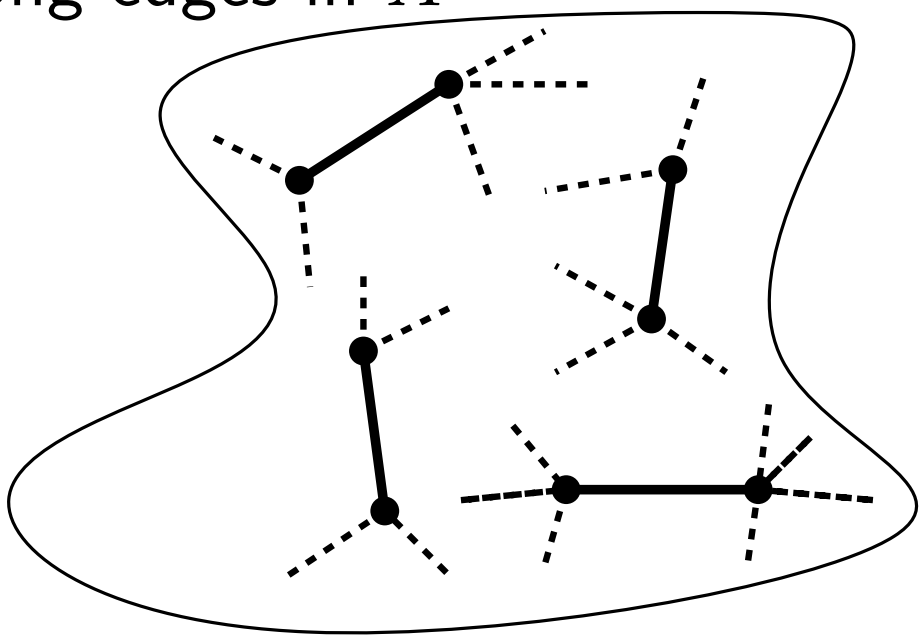

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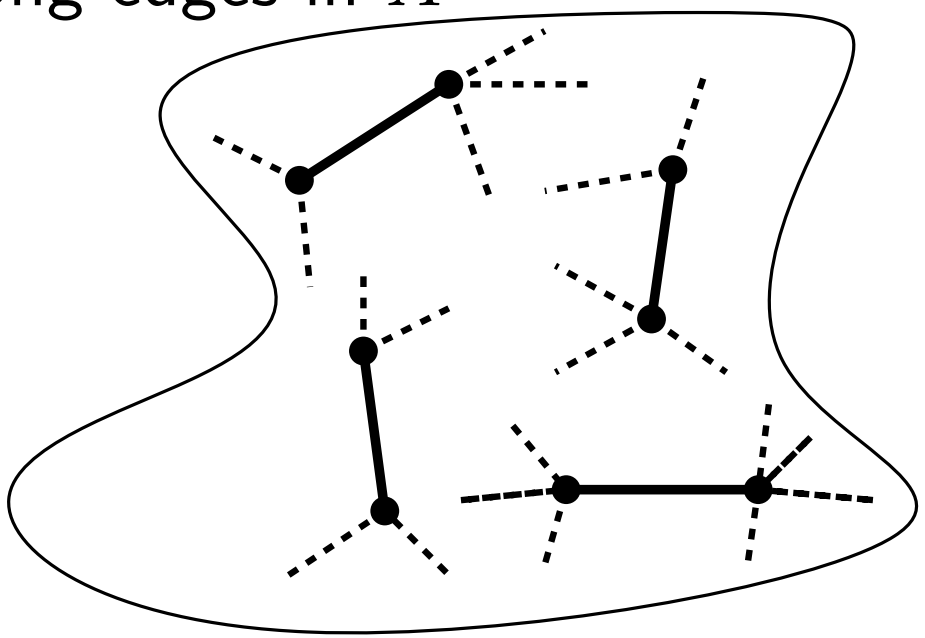

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Hence,

$$|C^*| \geq |A| = |C|/2 \Longrightarrow \frac{|C|}{|C^*|} \leq 2.$$

Reflection and methodology

How can we prove $C/C^* \leq 2$ when we don't know C^*?

Answer: By proving $C \leq 2|A|$ and $|A| \leq C^*$.

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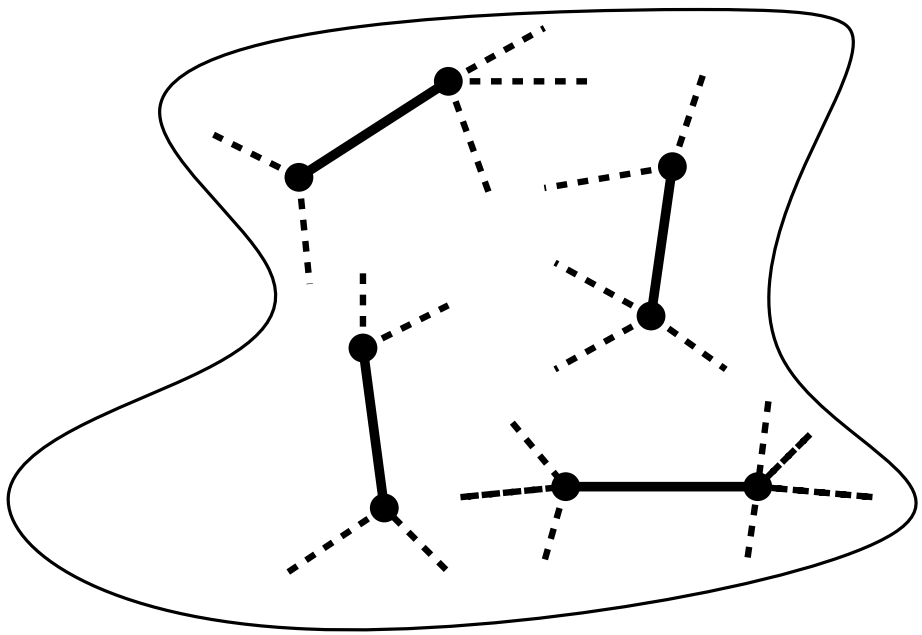
Answer: By proving $C \leq 2|A|$ and $|A| \leq C^*$.

General technique: Find a parameter $\square$ such that $C \leq \rho \cdot \square$ and $\square \leq C^*$.

For vertex cover: $\square = |A|$ and $\rho = 2$.

Question

Try to guess: Is there an approximation algorithm with a better approximation ratio?



History

1972: Karp's 21 NP-complete problems
(including vertex cover, set cover, Hamiltonian cycle and subset sum)

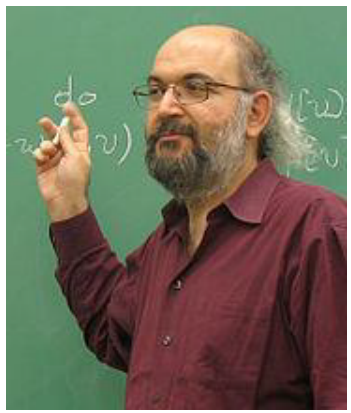
Karp

Turing Award

History

19xx: Many $\leq 2 - o(1)$.

Gavril



Yannakakis

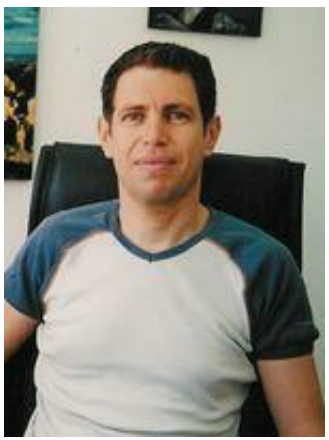
...

History

Assuming P≠NP:
1999: Håstad, $\geq 7/6$
2005: Dinur & Safra, ≥ 1.38
2018: Khot, Minzer, Safra, ≥ 1.41


Håstad


Dinur


Safra


Khot


Minzer

History

2008: Khot & Regev, $\geq 2 - \varepsilon$ assuming the Unique Game Conjecture.

Some, but not all people believe it.

Khot

Nevanlinna prize 2016

Regev

Traveling Salesperson

Given a complete undirected graph $G = (V, E)$.

For all $u, v \in V$, we are given $c(uv) \in \{0, 1, \ldots\}$.

Goal: Find minimum weight cycle through all vertices.

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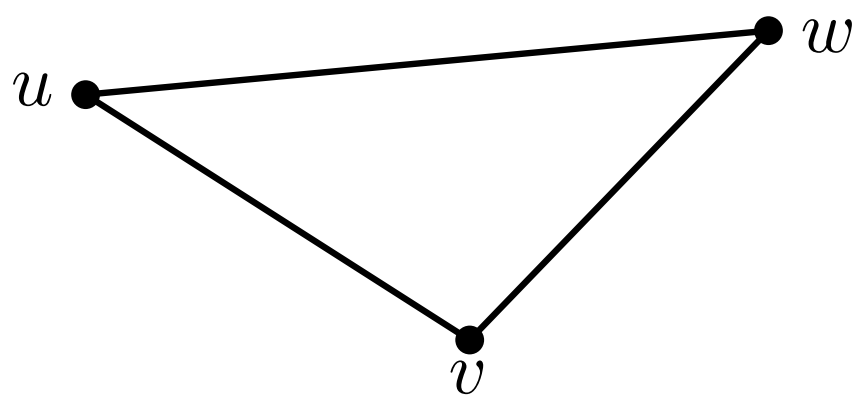
Assume: Triangle inequality: $c(uw) \leq c(uv) + c(vw)$.



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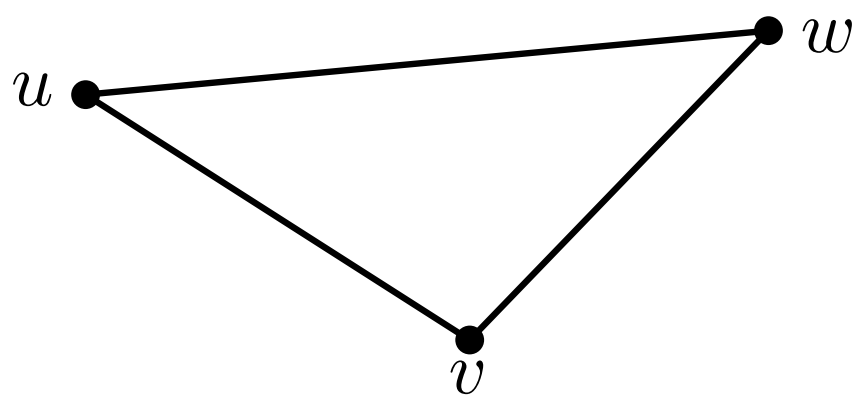
Still NP-hard!



Algorithm

APPROX-TSP(G, c)
- Find MST T
- Make Euler tour W using each edge of T twice
- Shortcut W to H by skipping duplicates
- Return H

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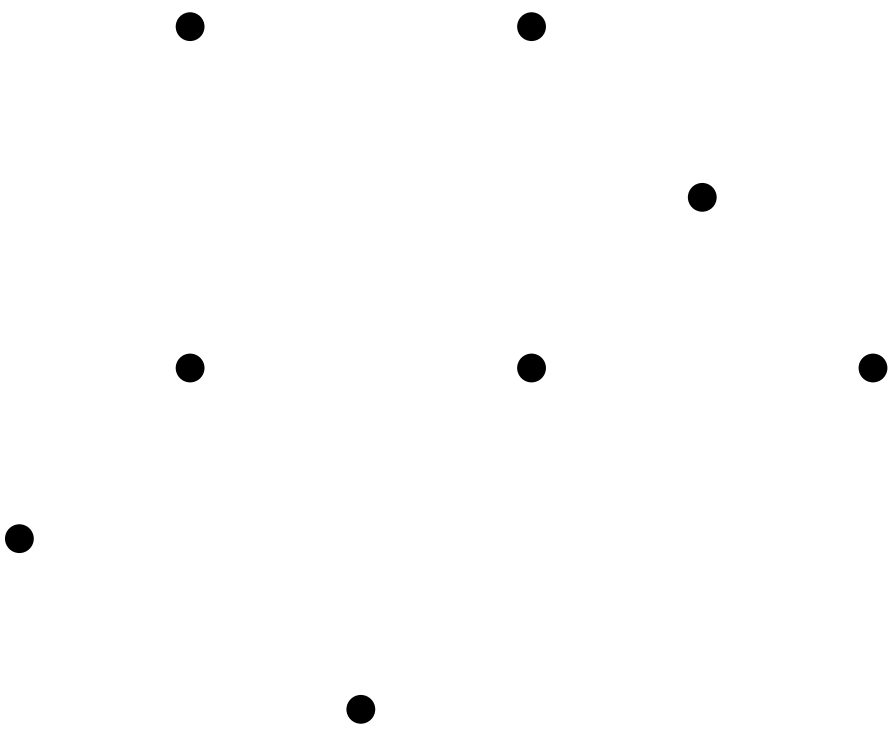

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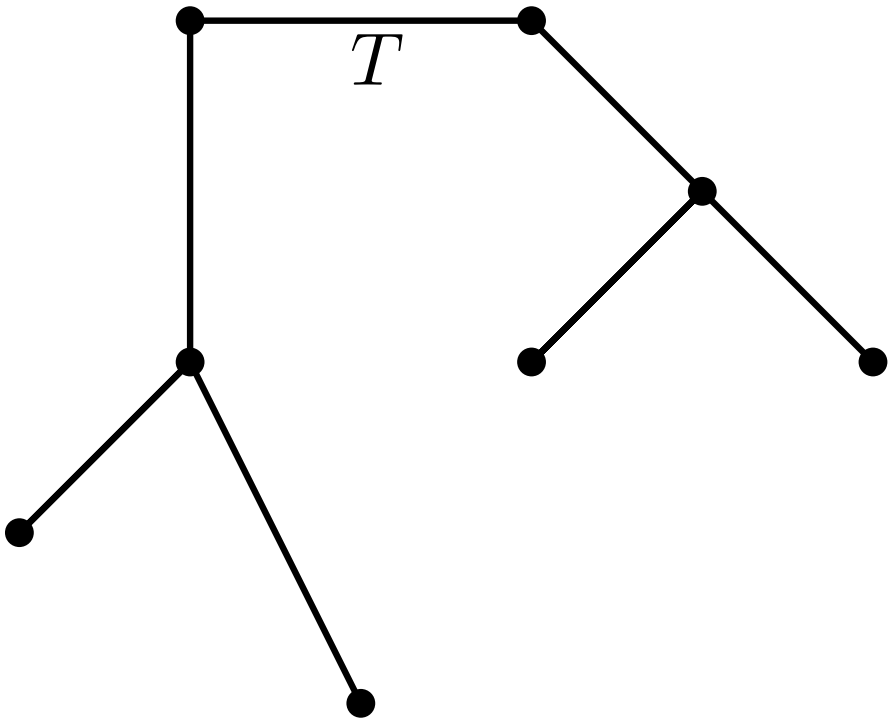

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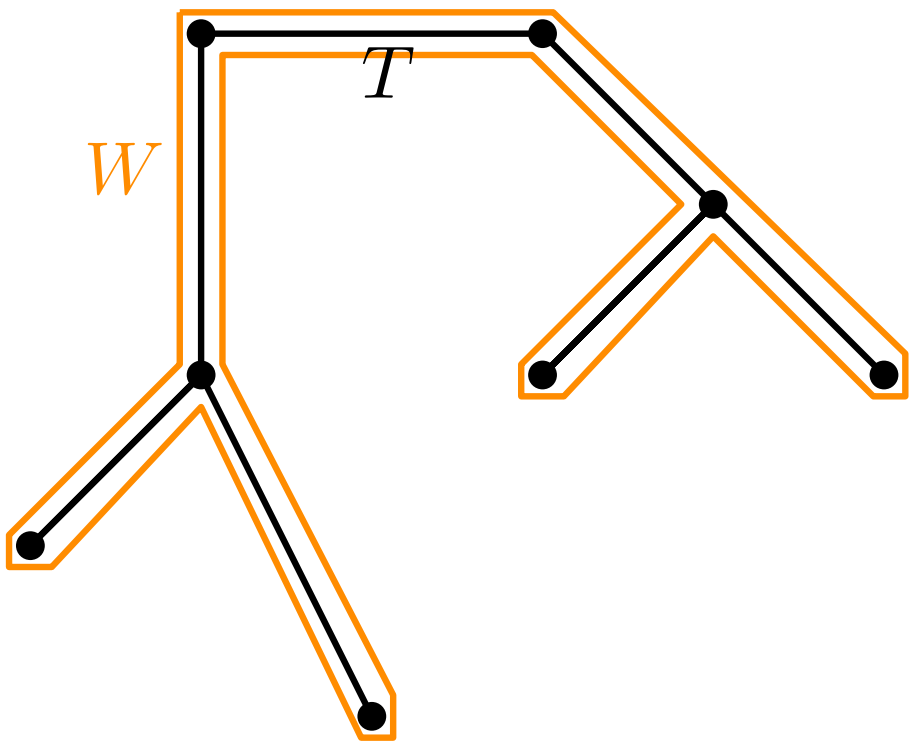

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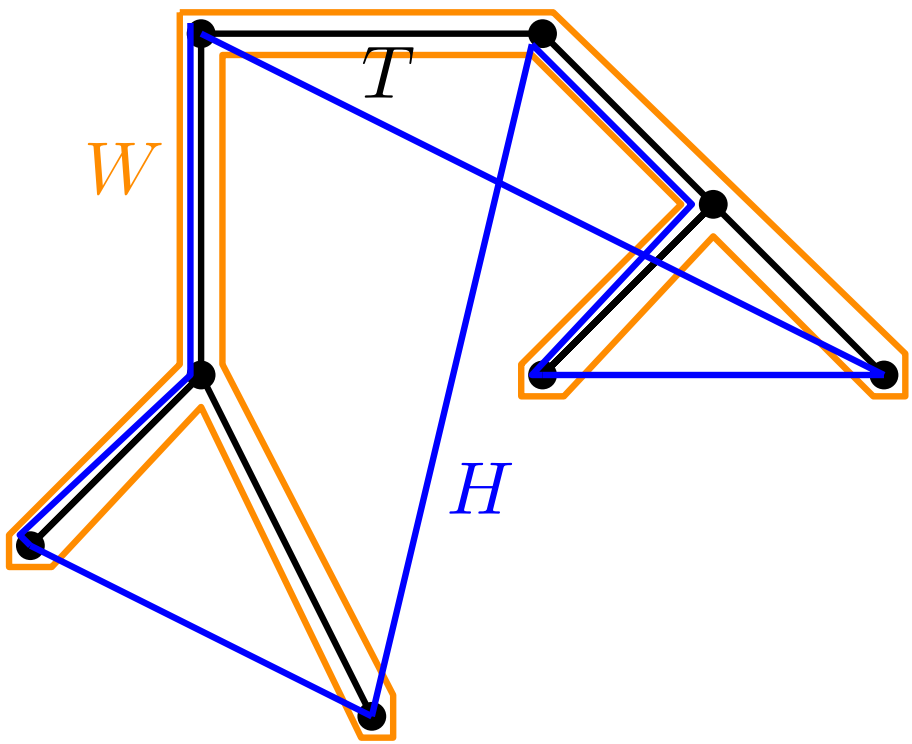

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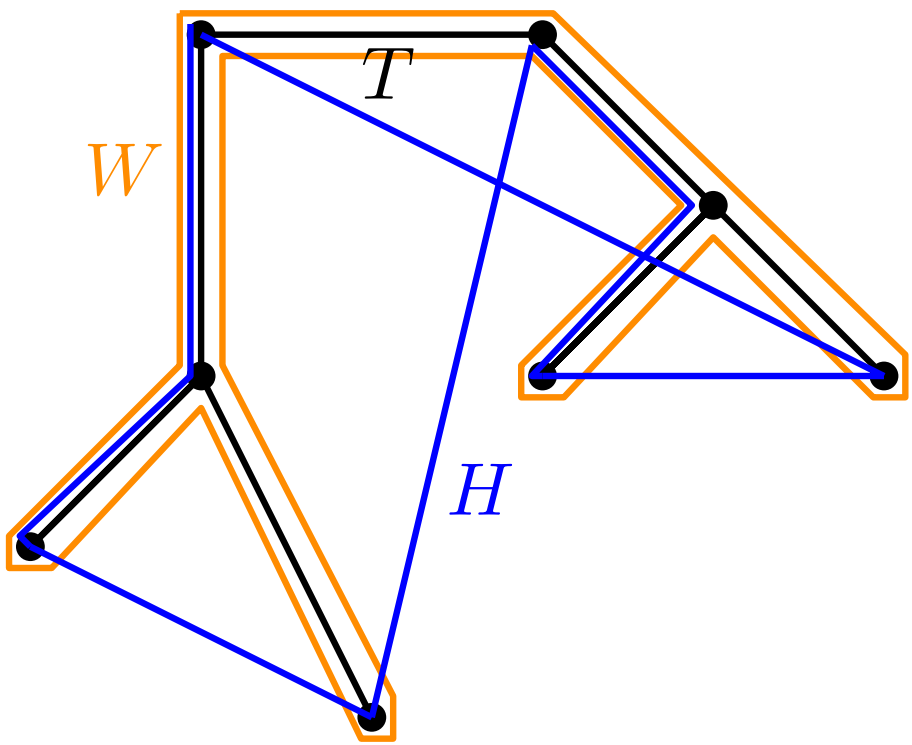

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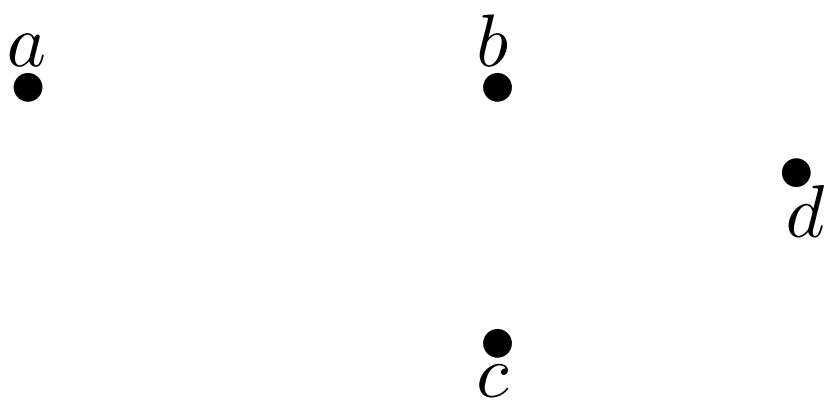

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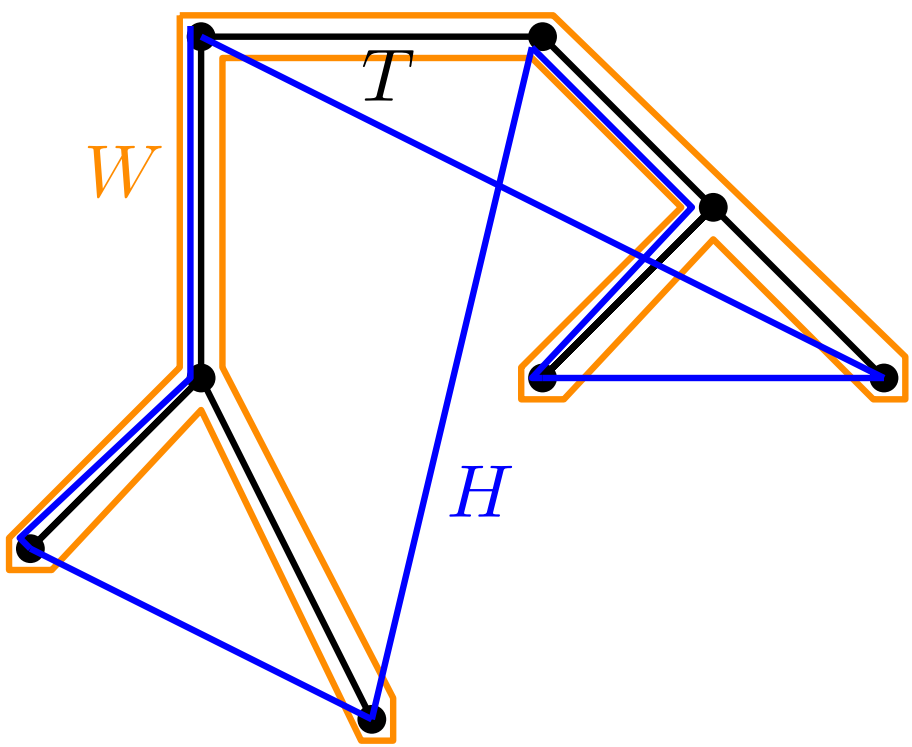

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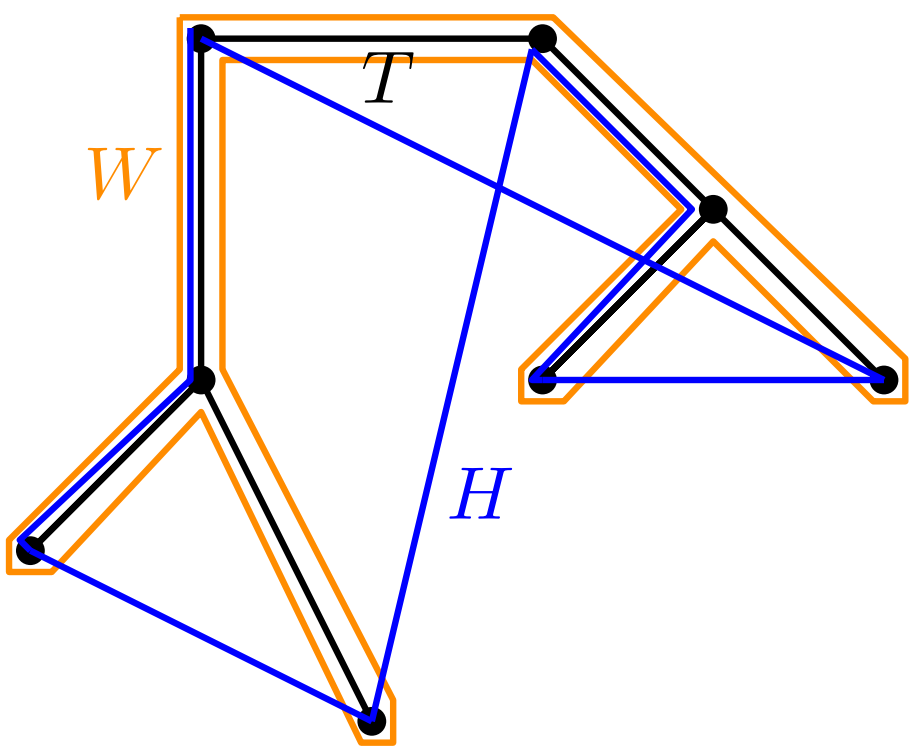

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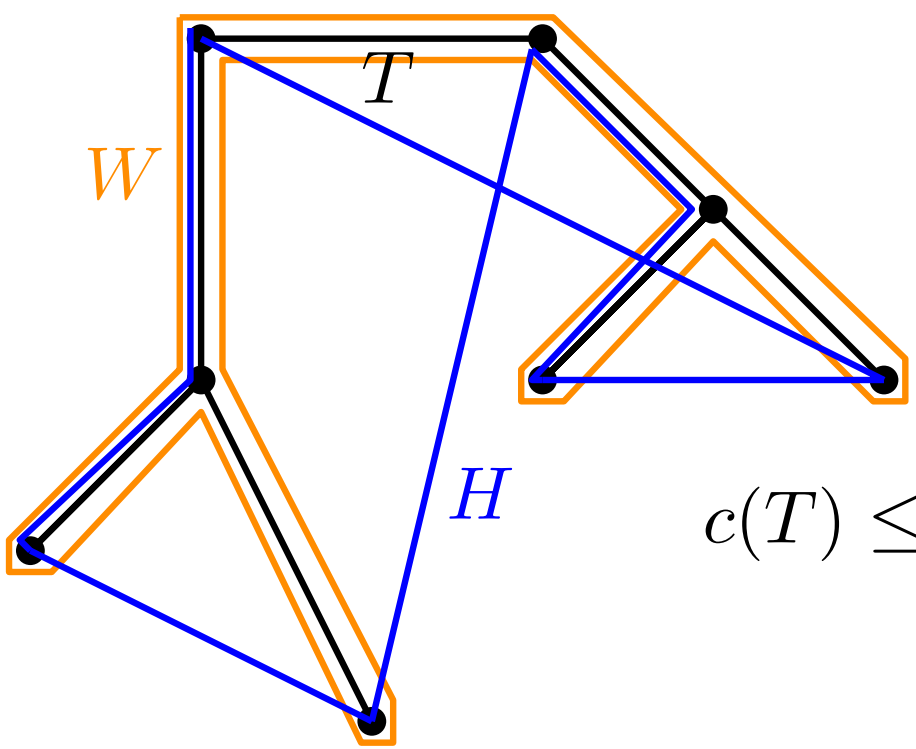

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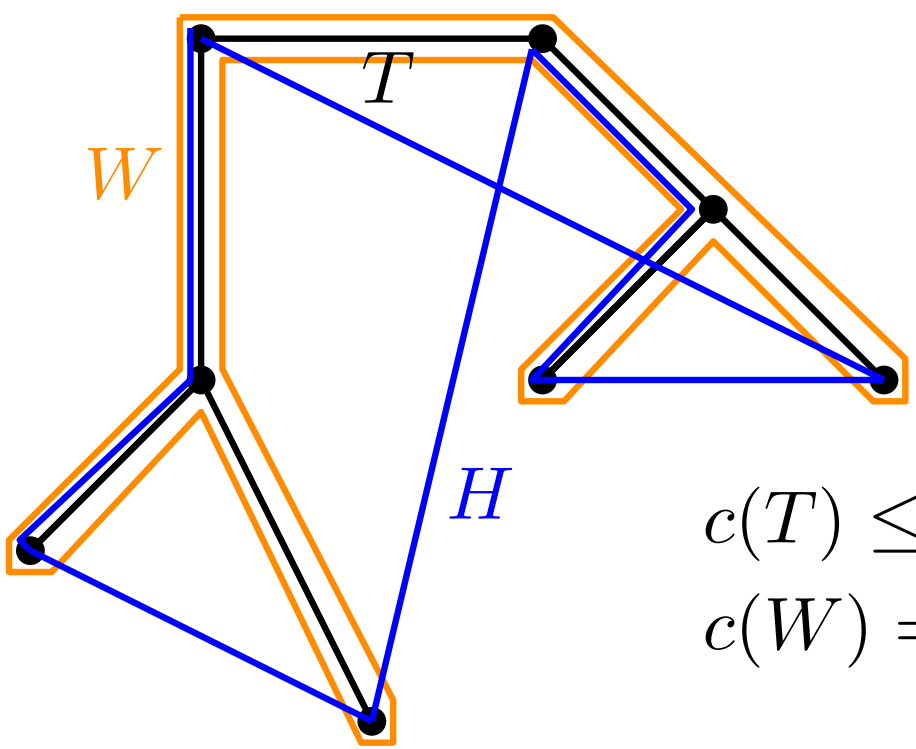

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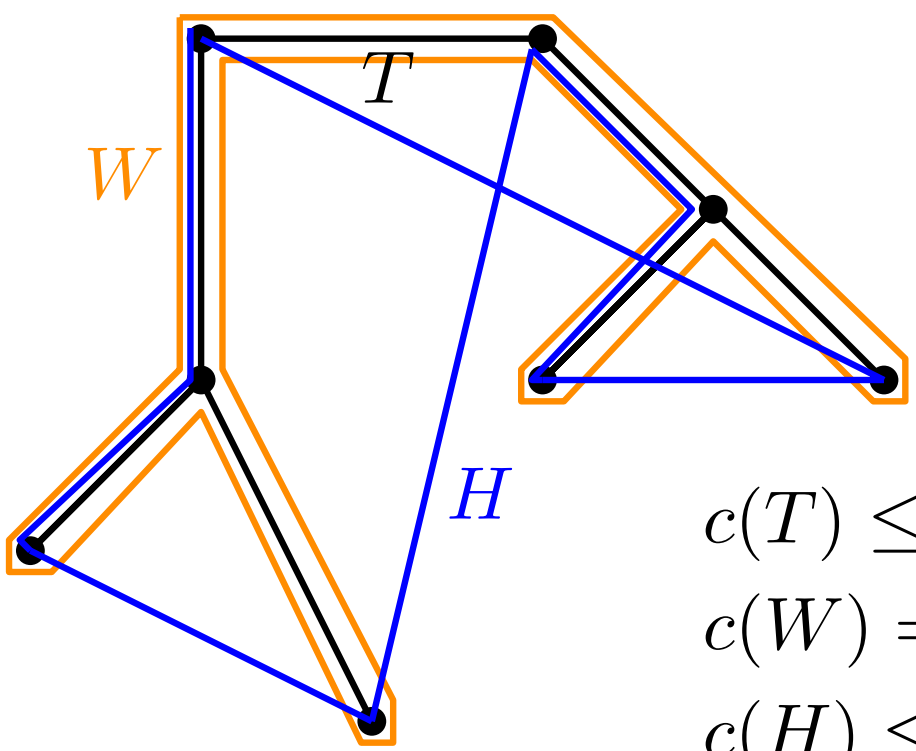

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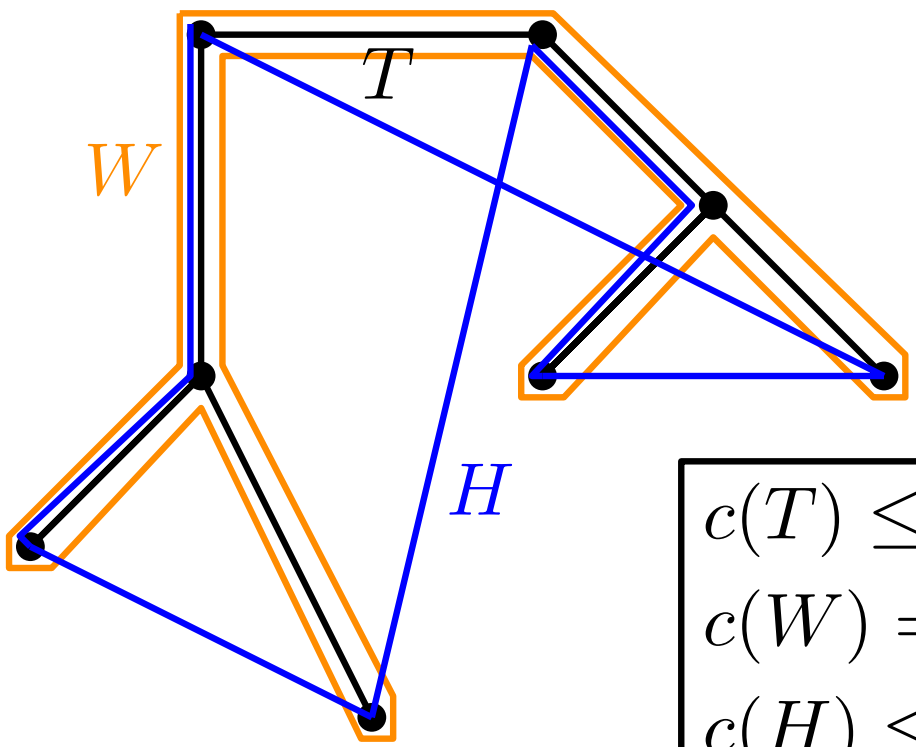

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Reflection and methodology

How can we prove $c(H)/c(H^*) \leq 2$ when we don't know H^*?

Answer: By proving $c(H) \leq 2c(T)$ and $c(T) \leq c(H^*)$.

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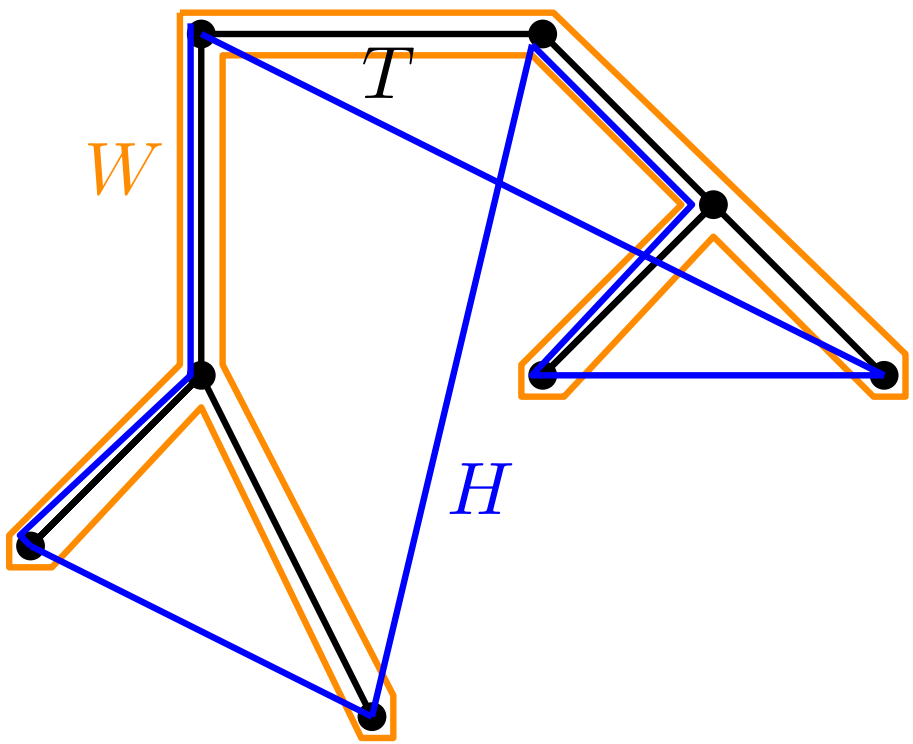
General technique: Find a parameter $\square$ such that $C \leq \rho \cdot \square$ and $\square \leq C^*$.

For TSP: $\square = c(T)$ and $\rho = 2$.

Question

Try to guess: Is there an approximation algorithm with a better approximation ratio?



History

1976: Christofides, Serdyukov, 1.5-apx algorithm
It's simple! See, e.g., Wikipedia. No improvement for decades

2021: Karlin, Klein, Gharan, $(1.5 - \varepsilon)$-apx algorithm for some $\varepsilon > 10^{-36}$

Quanta magazine

Computer Scientists Break Traveling Salesperson Record

24 |

After 44 years, there's finally a better way to find approximate solutions to the notoriously difficult traveling salesperson problem.

Set Cover

Input: Pair $(X, \mathcal{F})$, where X is a finite set and $\mathcal{F} \subseteq \mathcal{P}(X)$ is a family of subsets of X.

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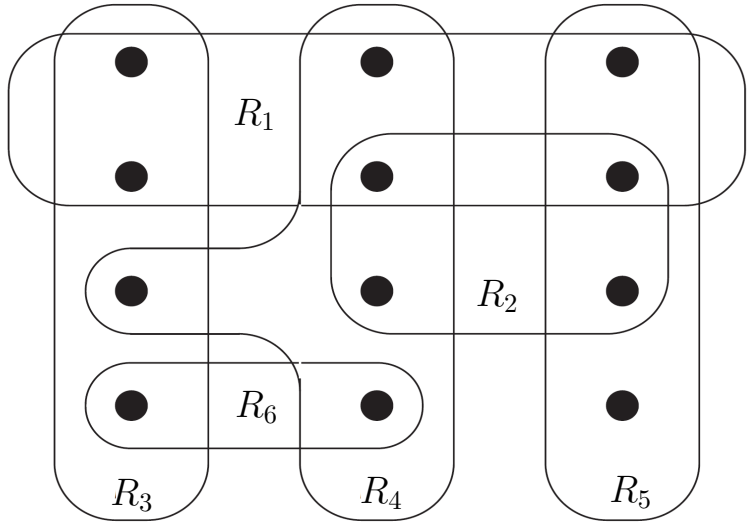

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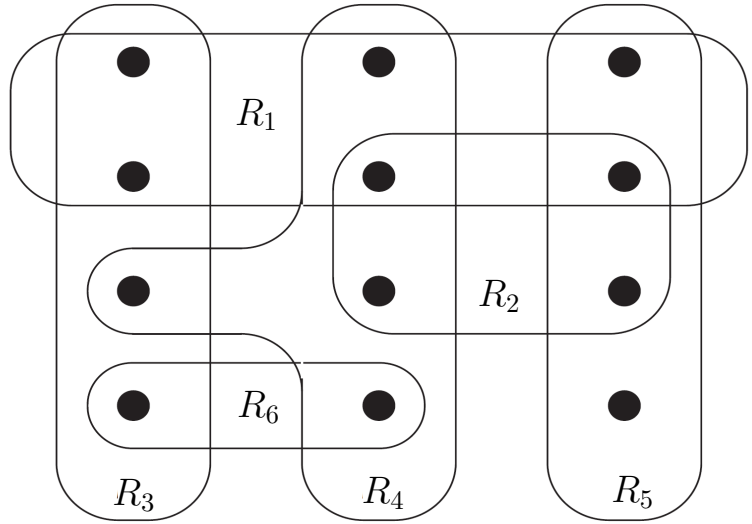
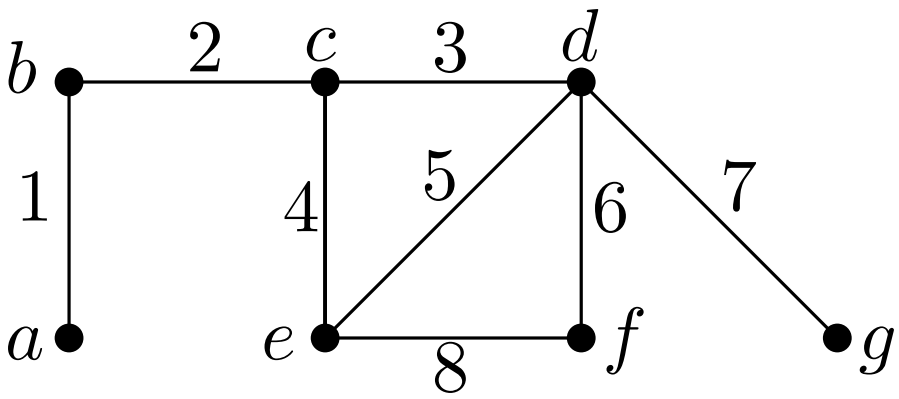
Exercise: Show that vertex cover is a special case.





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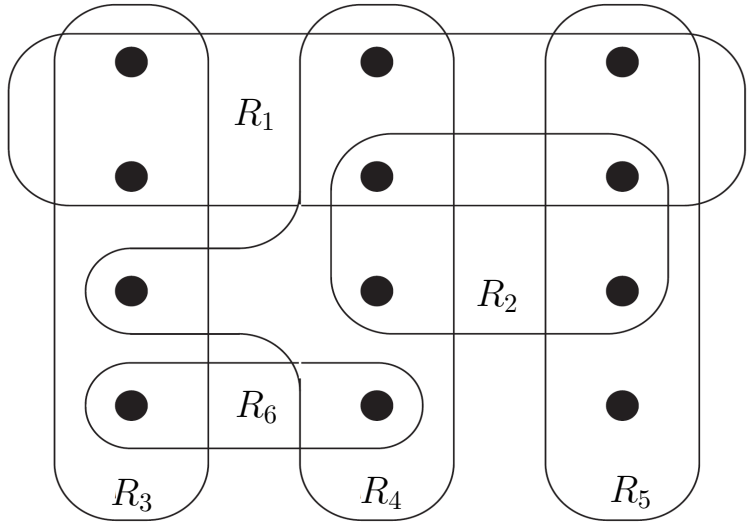
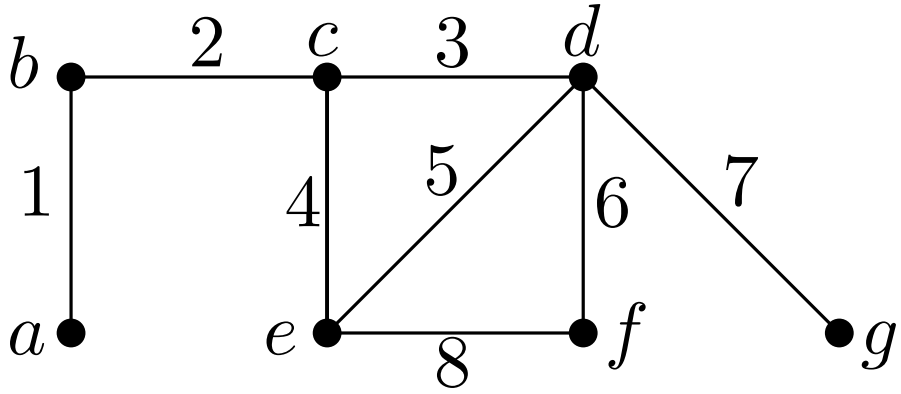


Exercise: Show that vertex cover is a special case.

$X := \{1, 2, \ldots, 8\}$

$\mathcal{F} := \{\{1\}, \{1, 2\}, \{2, 3, 4\}, \{3, 5, 6, 7\}, \{4, 5, 8\}, \{6, 8\}, \{7\}\}$



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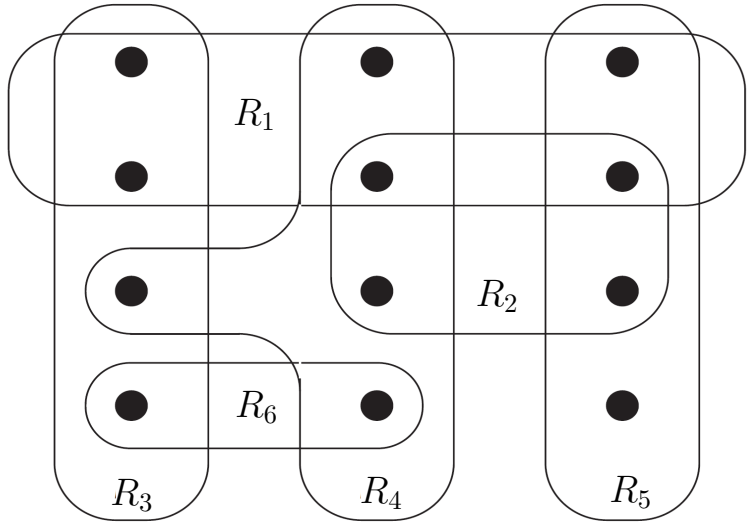

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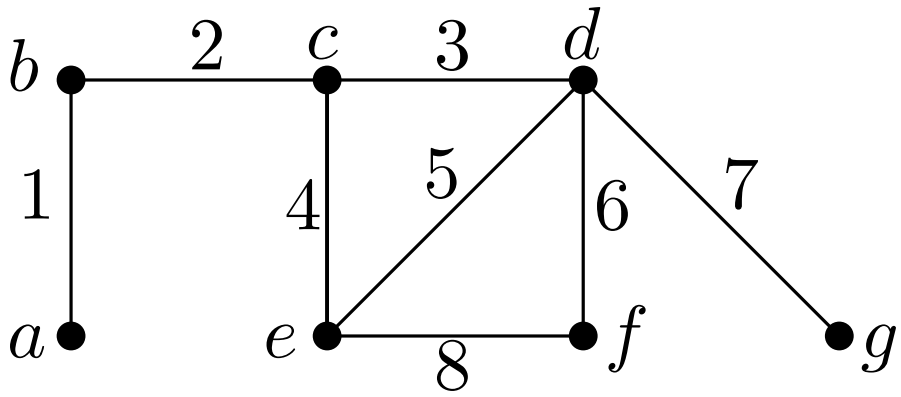

$X := E$

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Greedy Algorithm

GREEDY-SET-COVER($X, \mathcal{F}$)

$i := 0$

while $X \setminus S_{<i+1} \neq \emptyset$

$i := i + 1$

Pick $S_i \in \mathcal{F}$ with max $|S_i \setminus S_{<i}|$

Return $\mathcal{C} := \{S_1, \ldots, S_i\}$

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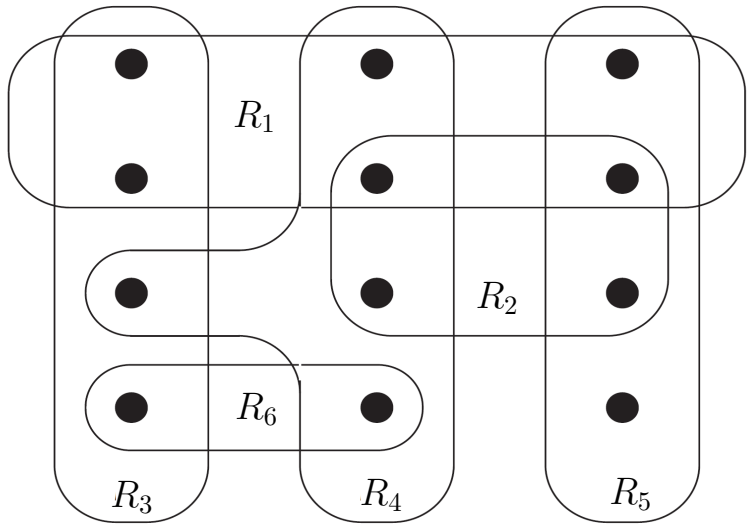

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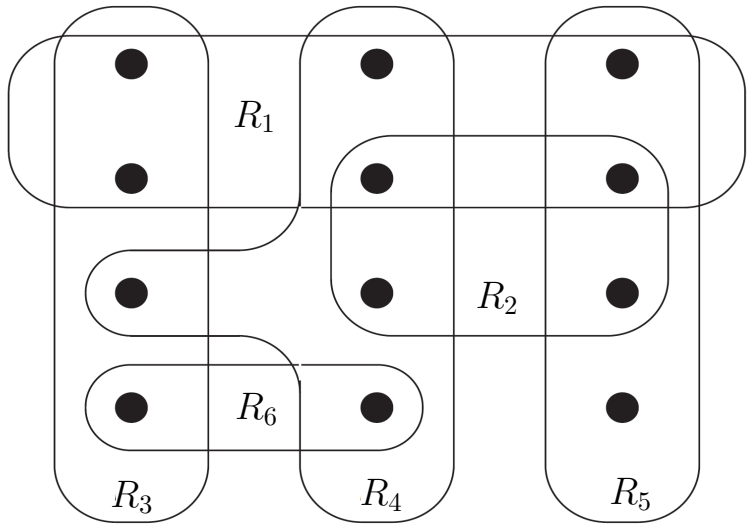

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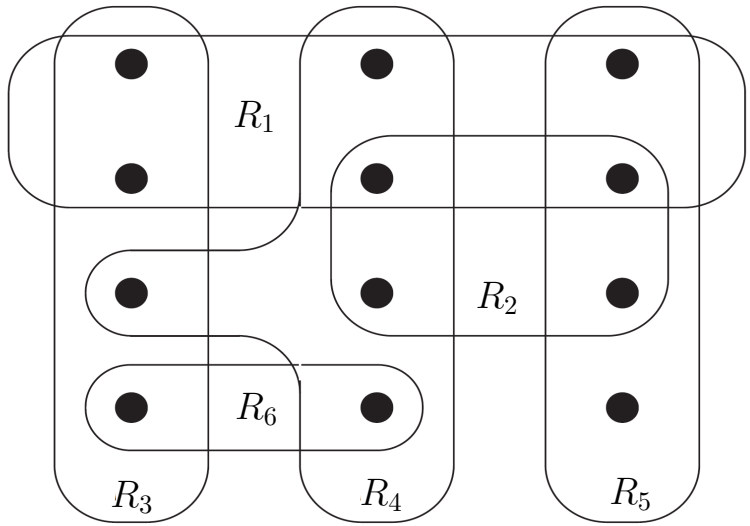
$S_1 := R_1$
$S_2 := R_4$



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$S_1 := R_1$
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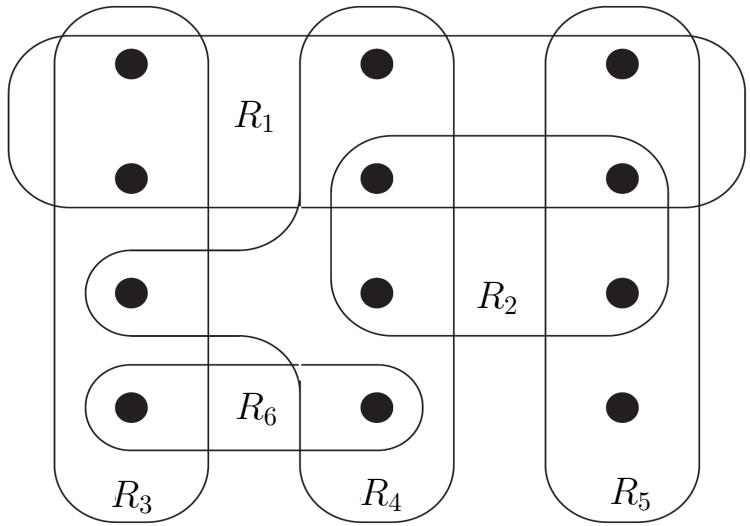

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$S_1 := R_1$
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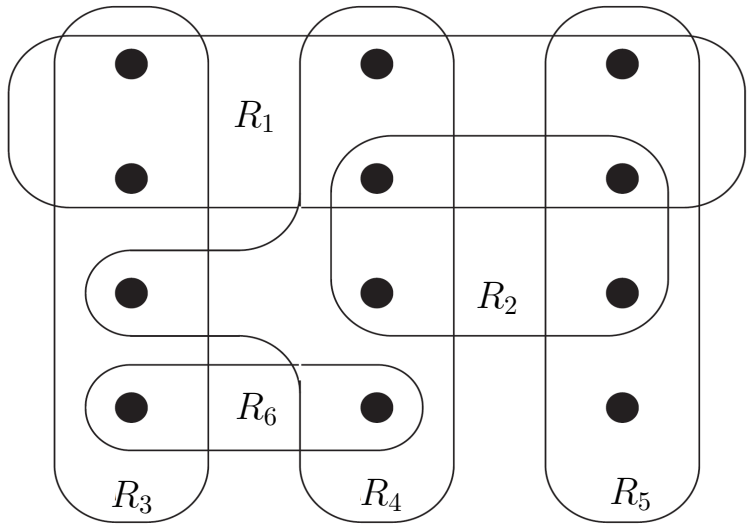
$S_4 := R_3$ or $S_4 := R_6$



Theorem

Thm.: For opt. sol. $\mathcal{C}^*$, we have

$$|\mathcal{C}| \leq H_{|X|} \cdot |\mathcal{C}^*|,$$

where

$$H_n := \sum_{i=1}^{n} 1/i \leq \ln n + 1.$$

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Hence, GREEDY-SET-COVER is a $O(\log n)$-approx. alg.

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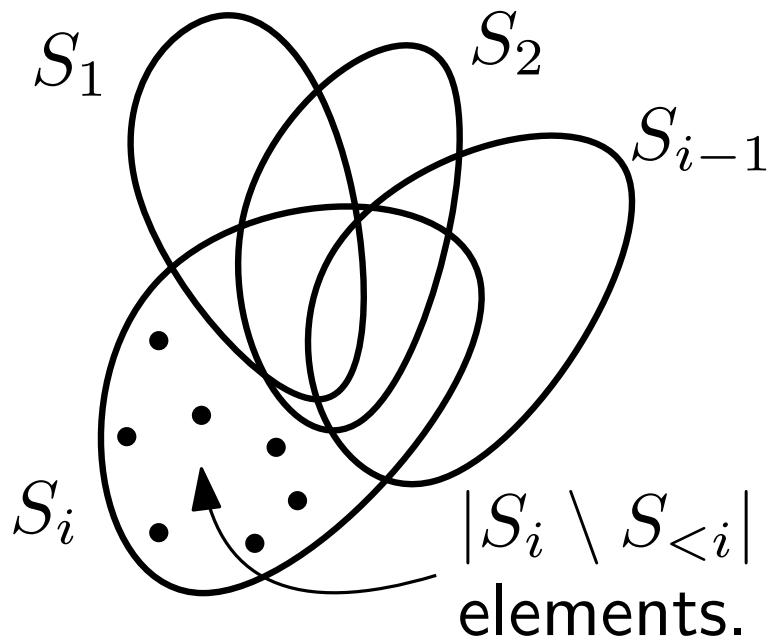

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$$c(X) = \sum_{i=1}^{|\mathcal{C}|} \sum_{x \in S_i \setminus S_{<i}} c_x = \sum_{i=1}^{|\mathcal{C}|} 1 = |\mathcal{C}|.$$

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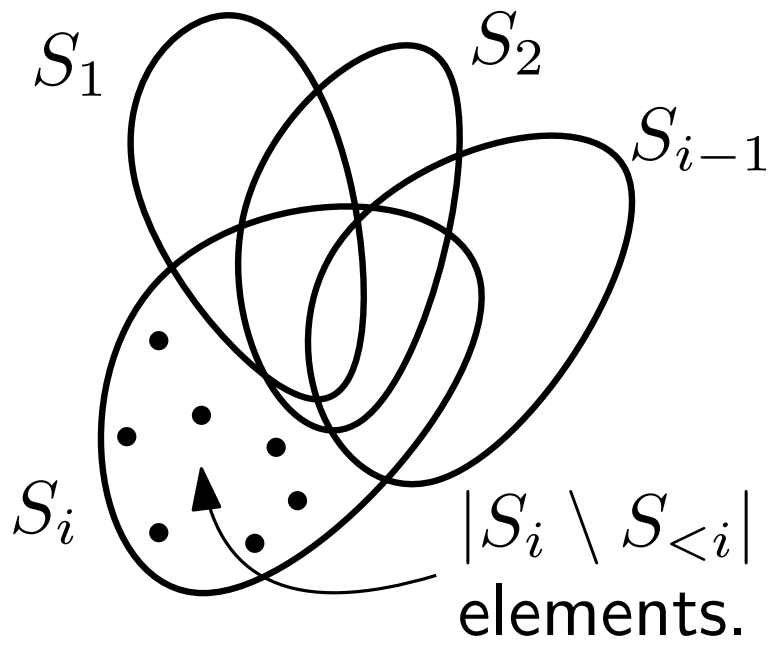

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Lemma: For all $S \in \mathcal{F}$:

$$c(S) \leq \sum_{i=1}^{|S|} \frac{1}{i} = H_{|S|}.$$

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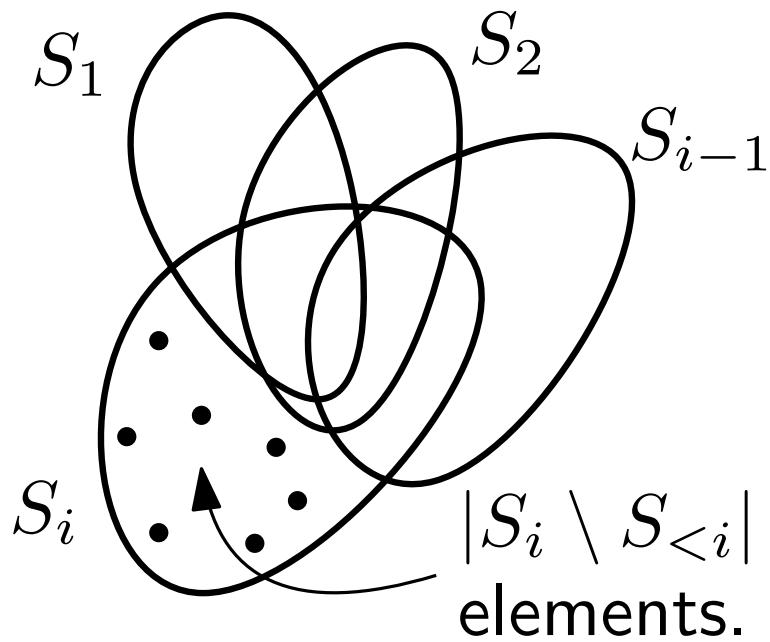

Theorem

Thm.: $|\mathcal{C}| \leq H_{|X|} \cdot |\mathcal{C}^*|$.

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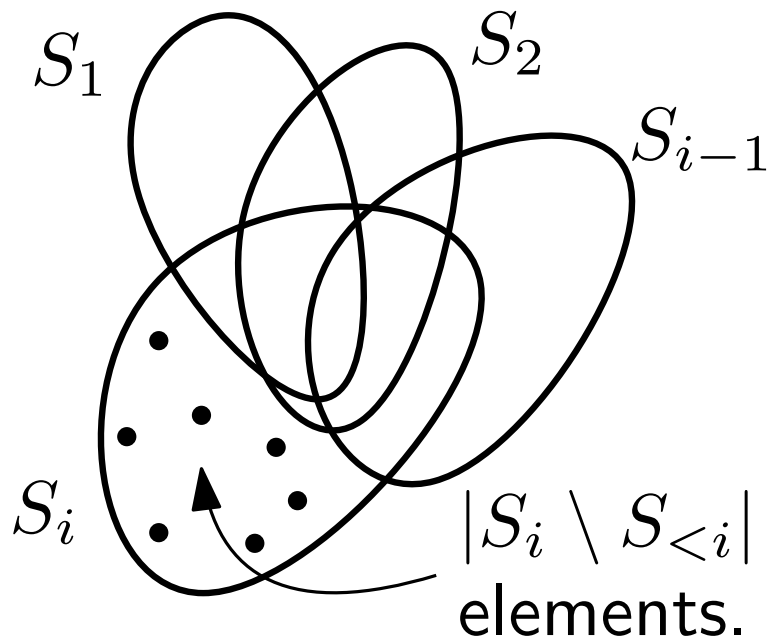
Observation:

$$c(X) = \sum_{i=1}^{|\mathcal{C}|} \sum_{x \in S_i \setminus S_{<i}} c_x = \sum_{i=1}^{|\mathcal{C}|} 1 = |\mathcal{C}|.$$

Lemma: For all $S \in \mathcal{F}$:

$$c(S) \leq \sum_{i=1}^{|S|} \frac{1}{i} = H_{|S|}.$$



Proof of Thm.:

$$|\mathcal{C}| = c(X) \leq \sum_{S \in \mathcal{C}^*} c(S) \leq \sum_{S \in \mathcal{C}^*} H_{|S|} \leq \sum_{S \in \mathcal{C}^*} H_{|X|} = |\mathcal{C}^*| \cdot H_{|X|}.$$

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Idea: 1st element in S to be covered has $c_x \leq \frac{1}{|S|}$, 2nd has $c_x \leq \frac{1}{|S|-1}$, ...

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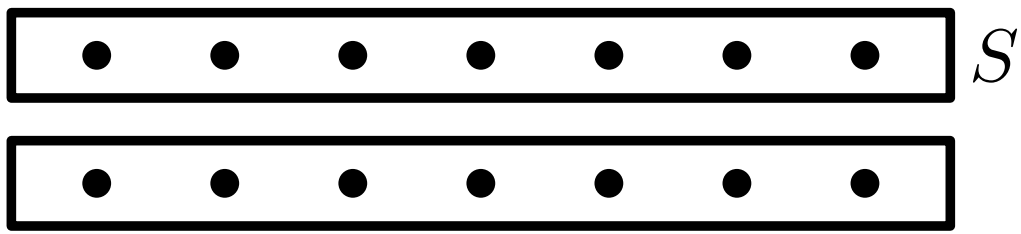

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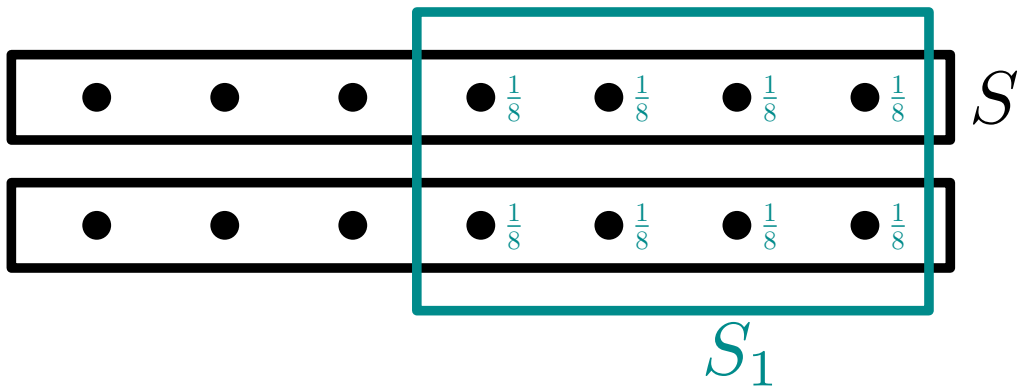

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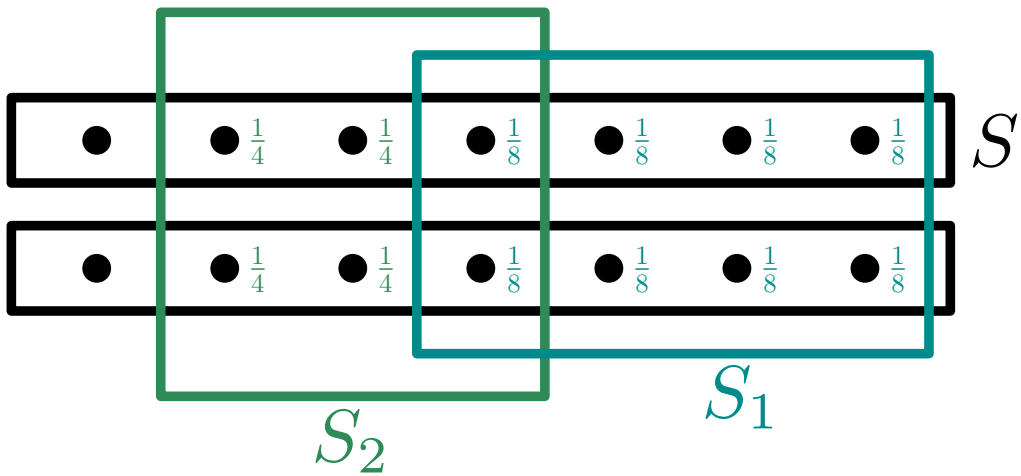

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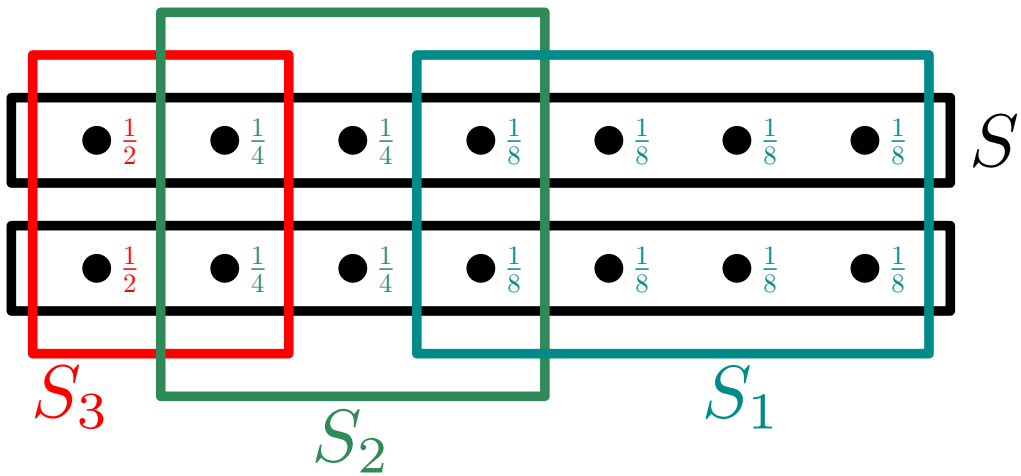

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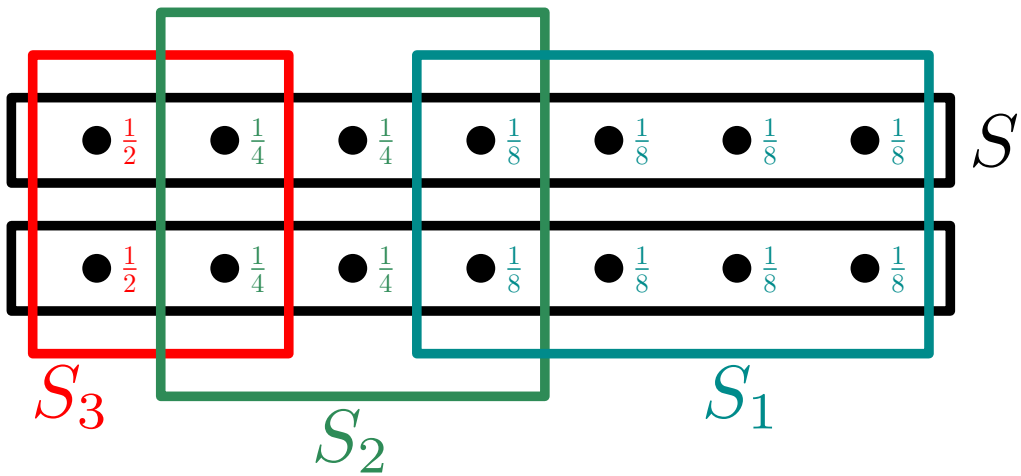

$$\begin{aligned} c(S) &= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &\leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \\ &= H_{|S|}. \end{aligned}$$

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$$c(S) = c_{x_1} + c_{x_2} + \ldots + c_{x_k} \leq 1 + \frac{1}{2} + \ldots + \frac{1}{k} = H_{|S|}$$

Using greedy algorithm for vertex cover

GREEDY-VERTEX-COVER(G)
  $C := \emptyset$
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The algorithm only gives a $\Theta(\log |E|)$-approximation.

Introduction to Approximation Algorithms, part II

13-1 2025, Srikanth Srinivasan,
DIKU (Slides: Mikkel Abrahamsen)

APPROX-SUBSET-SUM(S, t, ε)
  $L'_0 = [0]$
  for $k = 1, \ldots, n$
    L'_k=MERGE-LISTS($L'_{k-1}, L'_{k-1} + x_k$)
    L'_k=TRIM($L'_k, \varepsilon/2n$)
    remove duplicates and elm.s $> t$
  return last(L'_n)

Definition

Def.: An algorithm for an optimization problem has *approximation ratio* $\rho(n)$ if for every input of size n,

$$\max\left\{\frac{C}{C^*}, \frac{C^*}{C}\right\} \leq \rho(n).$$

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$C := \mathbf{E}\left[\text{cost(produced sol.)}\right]$

3-SAT

$$
\begin{aligned}
&(x_1 \vee x_7 \vee \neg x_9) \\
\wedge&(\neg x_7 \vee x_8 \vee x_9) \\
\wedge&(\neg x_2 \vee x_3 \vee \neg x_4) \\
&\vdots
\end{aligned}
$$

3-SAT

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Each clause has three *literals*
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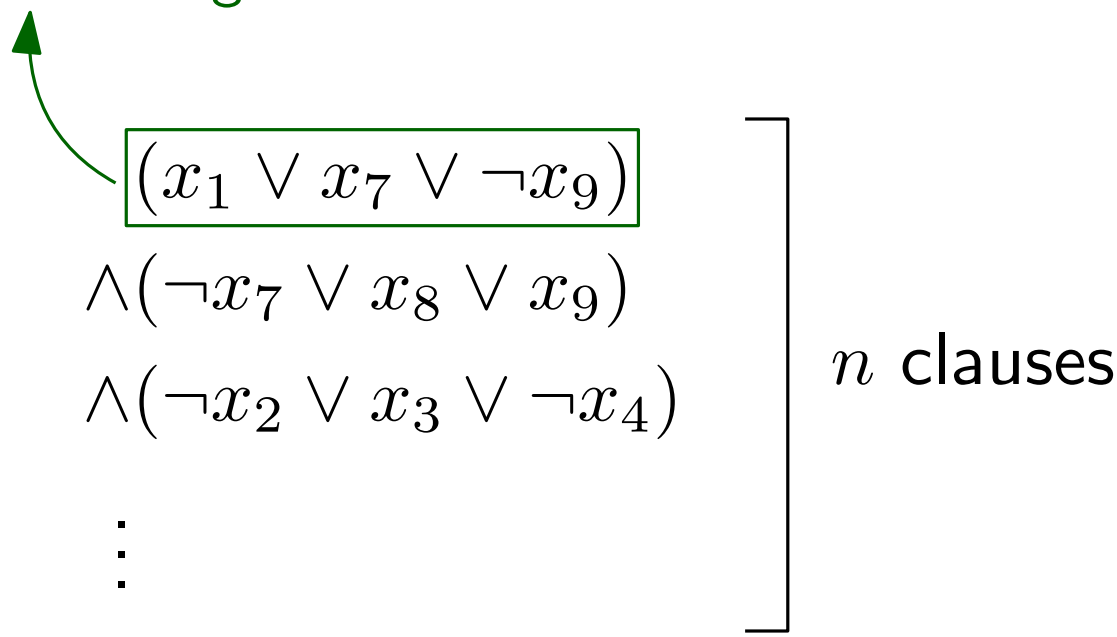

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n clauses

MAX-3-SAT: Find assignment that maximizes the number of true clauses.

Randomly assigning values

Φ is a MAX-3-SAT instance

RANDOM-ASSIGNMENT(Φ)
  for each variable x_i of Φ
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  return assignment

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$X := \sum_{i=1}^{n} [C_i] = \#$satisfied clauses

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Randomly assigning values

Φ is a MAX-3-SAT instance

RANDOM-ASSIGNMENT(Φ)
  for each variable x_i of Φ
    choose $x_i \in \{0, 1\}$ by flipping fair coin
  return assignment

Thm.: RANDOM-ASSIGNMENT is a $8/7$-approximation algorithm.

Proof: Let $\Phi = C_1 \wedge \ldots \wedge C_n$. Consider $C_i = \ell_1 \vee \ell_2 \vee \ell_3$.

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Method of conditional probabilities

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Bonus info

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By work of Håstad, it is NP-hard to approximate within $8/7 - \varepsilon$ for all $\varepsilon > 0$, so this very simple algorithm is essentially optimal, unless P=NP.

Vertex Cover

Def.: Let $G = (V, E)$ be a graph. A set $V' \subseteq V$ of vertices is a *vertex cover* if for all $uv \in E$, we have $u \in V'$ or $v \in V'$.

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APPROX-VERTEX-COVER(G)

$C := \emptyset$

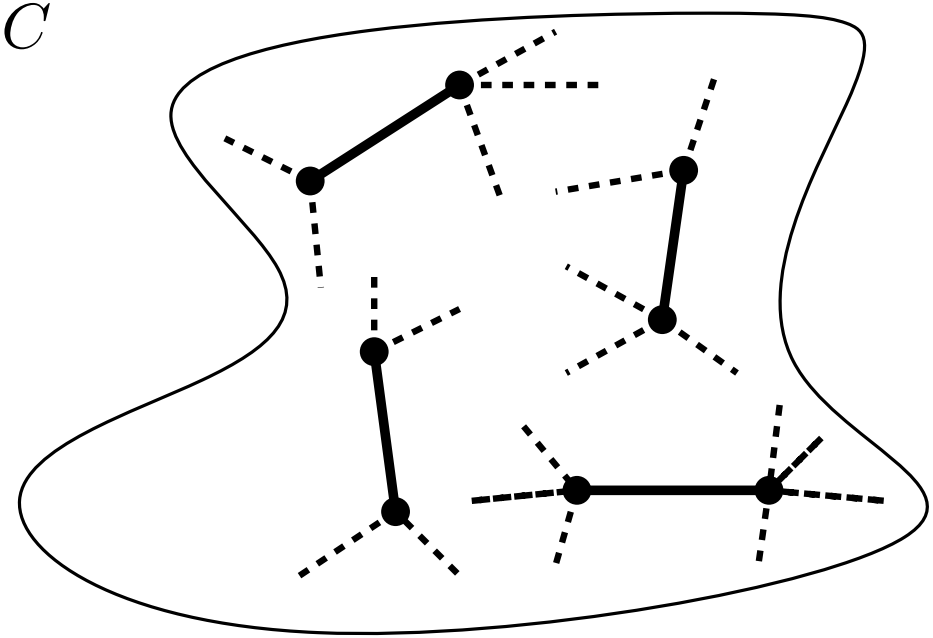
while $E(G) \neq \emptyset$

choose $uv \in E(G)$

$C := C \cup \{u, v\}$

remove all edges incident on u or v from $E(G)$

return C



$$\frac{C}{C^*} \leq 2$$

Weighted Vertex Cover

Def.: Let $G = (V, E)$ be a graph. A set $V' \subseteq V$ of vertices is a *vertex cover* if for all $uv \in E$, we have $u \in V'$ or $v \in V'$.

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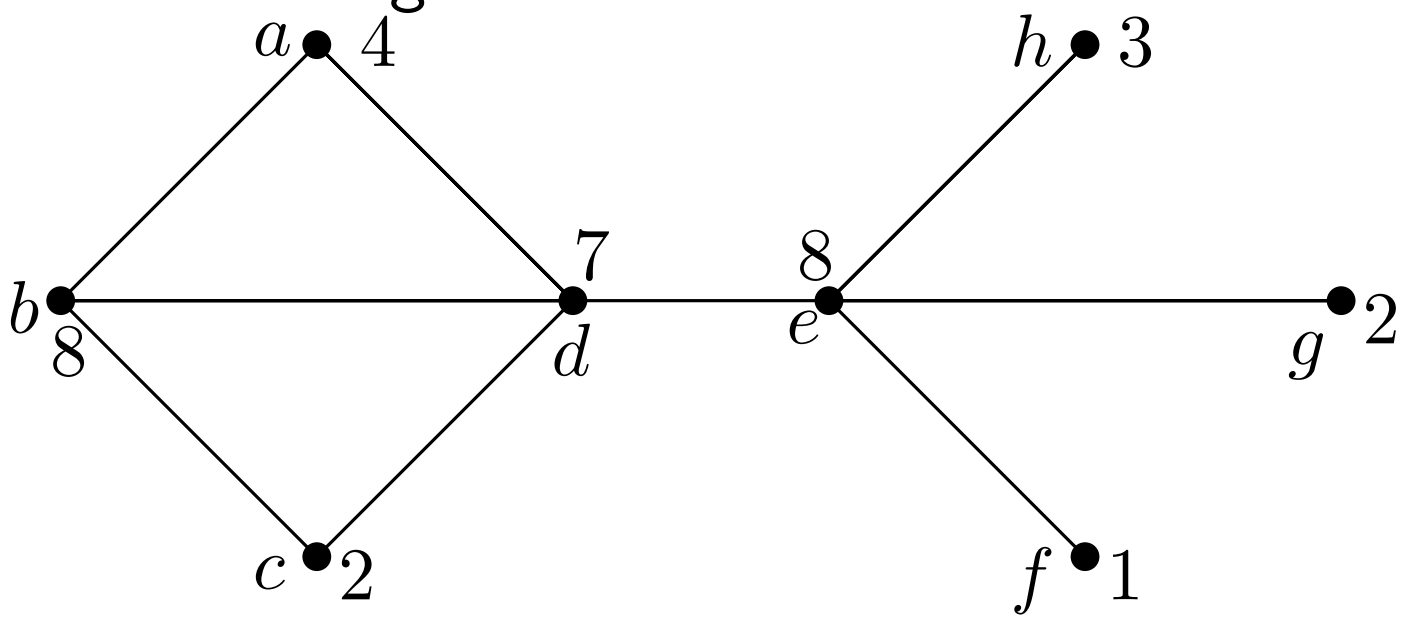
Exercise: Find minimum (unweighted) vertex cover and then minimum weighted vertex cover.



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0-1-integer program (IP):

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$x_u + x_v \geq 1, \forall uv \in E$ $\quad$ (edge uv covered)

$$\text{minimize} \sum_{v \in V} w(v) x_v$$

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Relaxed solution can be smaller, not larger

Algorithm

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2) $uv \in E \Rightarrow \bar{x}_u + \bar{x}_v \geq 1 \Rightarrow \bar{x}_u \geq \frac{1}{2} \vee \bar{x}_v \geq \frac{1}{2} \Rightarrow u \in C \vee v \in C$. ✓

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minimize $\sum_{v \in V} w(v) x_v$

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Compute opt. sol. $\bar{x}$ to LP

return $C := \{v \in V \mid \bar{x}_v \ge 1/2\}$

Need to prove: 3) $\frac{w(C)}{w(C^*)} \le 2$.

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$$\le 2 \sum_{v \in V} w(v)\bar{x}_v = 2z^* \le 2w(C^*) \Rightarrow \frac{w(C)}{w(C^*)} \le 2. \quad \checkmark$$

Reflection and methodology

How can we prove $w(C)/w(C^*) \leq 2$ when we don't know $w(C^*)$?

Answer: By proving $w(C) \leq 2z^*$ and $|z^*| \leq w(C^*)$.

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General technique: Find a parameter $\square$ such that $C \leq \rho \cdot \square$ and $\square \leq C^*$.

For weighted vertex cover: $\square = z^*$ and $\rho = 2$.

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Claim: $(1+\delta)^n \leq 1 + 2n\delta$ if $2n\delta \leq 1$.

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Total running time: $O\left(\sum_{k=1}^{n} |L'_k|\right) = O\left(\frac{n^2 \ln t}{\varepsilon}\right)$.